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Basic Statistical Properties of the Knot Efficiency

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Abstract: A knot is the weakest point of every rope, and the *knot efficiency* measures the portion of original rope strength taken away by the knot. Despite possible safety implications, surprisingly little attention has been paid to this life-critical quantity in research papers. Knot efficiency is directly immeasurable and the only way to obtain it is by calculation from rope breaking strength. However, this complication makes room for a wide spectrum of misleading concepts. The vast majority of authors do not treat knot efficiency as a random variable, and published results mostly suffer from incorrect statistical processing. The main goal of the presented paper is to fix this issue by proposing correct statistical tools needed for knot efficiency assessment. The probability density function of knot efficiency $\psi(\eta)$ has been derived in general, as well as for normally distributed breaking strength. Statistical properties of knot efficiency PDF have been discussed, and a less complex approximation of knot efficiency PDF has been proposed and investigated.

Keywords: knot efficiency; probability density function; cumulative distribution function; statistical properties

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1. Introduction

Mountain rescuer, mountain guide, alpine climber, speleologist, worker at height—these are professions where ropework and knot manipulation can be considered as life-critical and must be mastered at a professional level. Despite possible safety implications, surprisingly little attention has been paid to so-called *knot efficiency* in scientific journals and research papers. Available textbooks and guidelines refer to the obsolete information sources. Many cited conclusions are based on weak statistical datasets, and tested knots are occasionally tied incorrectly, paying minimal attention to the type of load and distinguishing symmetry of knots. Experimental setup has usually not been professionally built, and regularly calibrated, incomparable studies are merged together to produce useless conclusions. The last but not least, incorrect statistical apparatus is used to evaluate the results and to express their interpretation [1,2]. The majority of mentioned issues have already been addressed in a recent study [2]. However, the problem with incorrect statistical treatment of knot efficiency remains. In the following text we would like to point to the common misleading ideas as the most common sources of statistical mistakes.

Experimental data has shown that knot efficiency cannot be treated as a sharply edged single value but as a continuous random variable [2,3]. Accepting this concept immediately introduces a set of questions: What is the probability distribution of knot efficiency? How can we find typical statistical benchmarks such as mean, variance, median, mode, and tolerance intervals? All these questions should be answered, and a consistent mathematical methodology for knot efficiency assessment should be proposed and established.

It is a well-known fact that a knot is the weakest point on a rope [4]. Therefore, it is not surprising that every rope breaks at the point where a knot is placed. To measure how much of original rope strength was taken away by placing a knot, the so-called *knot efficiency* $\eta = \frac{X}{Y}$ has been introduced [1–3].

Definition 1. *Let X be the static breaking strength of rope where a single particular knot has been tied, and let Y be the straight rope static breaking strength, both measured in units of force. We say that dimensionless quantity*

$$\eta = \frac{X}{Y} \tag{1}$$

is the knot efficiency of a particular knot.

Knot efficiency is usually expressed in percent. With probability bordering with certainty, a knotted rope is weaker than a straight one; hence, the knot efficiency is a number between 0 and 100%. As a rule of thumb we can say that the higher the knot efficiency, the higher the probability that particular knot withstand loaded force. For example, there is a group called loop knots, designated mainly to form an anchoring loop. It has been shown that efficiency of loop knots ranges approximately from 45% to 85%, depending on a particular knot, its symmetry, and the geometry of applied forces [1–3,5–7].

Based upon the Definition 1 it is clear that no one can evaluate knot efficiency by a single measurement. It is necessary to measure breaking strength of the knotted rope firstly, and subsequently to select another piece of rope from the same batch and measure straight rope breaking strength. Then, it is possible to calculate the ratio.

Statistical analysis performed on experimental dataset of broken ropes showed that both the quantity X as well as Y can be considered as a random variable with normal probability distribution function. Approximately 200 straight rope breaking strengths measurements and 80 knotted rope breaking strength measurements have been studied using Q-Q plots, Kolmogorov-Smirnov and Shapiro-Wilk tests. The target was to check the null hypothesis, i.e., *the normal distribution model fits the observations*. The results did not show a significant departure from the normality; therefore, the null hypothesis could not be rejected [2]. This immediately leads to the conclusion that heterogeneity is a fundamental feature of every rope, breaking strength is different from point to point and its variance cannot be neglected. It is crucial to become familiar with concept that any value not forbidden by physical laws can be measured with some probability and the same breaking strength will not always be observed.

At this point, the problem of knot efficiency evaluation become challenging for many authors. Two misleading concepts can be found across the textbooks and research papers:

1. Two randomly selected pieces of rope are picked. A knot is placed on the first rope segment and afterwards it is broken to get value X. A second rope segment is broken in a straight setup without placing a knot to get value Y. Knot efficiency is calculated as the ratio $\eta = \frac{X}{Y}$. The total number of performed measurements is 2.
   This basic approach is the most widely spread because it is simple, relatively fast, and inexpensive (only two pieces of rope need to be destroyed). However, at the same time, results are significantly dependent on rope segment selection because this strategy does not take heterogeneity of inspected rope into account. Risk of improper efficiency assessment is high. Although this may seem obvious for the educated reader, this is the most common mistake, widely spread across the vast majority of texts.
2. A more advanced, but still incorrect approach is based on the following idea. Split a rope into $n + m$ segments (unfortunately both n and m are usually a small numbers). Place a particular knot on randomly picked n of them, and perform breaking strength measurements. Get the experimental outcomes $D_X = \{x_1, \ldots, x_n\}$ and evaluate mean value $E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i$, and sometimes also the variance $Var(X) = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - E(X))^2$ is evaluated. Break the remaining m segments in a straight setup without placing a knot to get the set of values $D_Y = \{y_1, \ldots, y_m\}$ with mean value $E(Y)$ and variance $Var(Y)$. Evaluate the mean knot efficiency as the ratio $E(\eta) = \frac{E(X)}{E(Y)}$. Variances are usually left aside as they are not recognised as being a useful piece of information.

This concept is the second most widely spread and slightly better than the first one since it at least reflects the heterogenous nature of rope and it implicitly indicates that the author is familiar with the random nature of breaking strength. However, it remains incorrect even if x and y are independent, at least because it is easy to prove that $\underbrace{E(X/Y) = E(X)E(Y^{-1})}_{x,y\text{- independent}} \neq E(X)\frac{1}{E(Y)} = \frac{E(X)}{E(Y)}$ [8].

The main objective of the presented paper is to shed light on the mathematical background of knot efficiency properties as a random variable. Four main objectives can be formulated:

- to derive the PDF of the knot efficiency in general (see Section 2.1);
- to derive the special case PDF $\psi(\eta)$ of the knot efficiency when breaking strength of knotted and straight rope is normally distributed (see Section 2.2);
- to explore properties of $\psi(\eta)$ (see Section 2.3);
- to find a simpler yet accurate approximation to a relatively complex function $\psi(\eta)$ in order to make calculations less demanding. In addition, we aim to specify the field of application, advantages, and drawbacks (see Section 2.4).

2. Results

2.1. Probability Density Function of the Knot Efficiency $\psi(\eta)$ in General

Quantities X and Y can be considered as absolutely independent by definition. Tested rope pieces are carefully shuffled before measurement, and the very nature of breaking strength measurement is destructive. Performing repeated measurement on the same piece of rope is ruled out. Hence we are able to write following relation:

$$P(a \leq X < b, c \leq Y < d) = P(a \leq X < b)P(c \leq Y < d). \quad (2)$$

We can even more generally write an equation for probability of such events where the measurement of the independent vector (X, Y) falls into a simple connected region $M \subset \mathbb{R}^2$ using a continuous joint probability density function $\rho(x, y) = \rho_X(x)\rho_Y(y)$ [8]:

$$P((X, Y) \in M) = \iint_M \rho_X(x)\rho_Y(y)\mathrm{d}x\mathrm{d}y. \quad (3)$$

Here, ρ_X is the PDF of knotted rope breaking strength and ρ_Y is the straight rope breaking strength PDF in general. Both of them are continuous for $\forall x, y \in \mathbb{R}$ and normalised: $\int_{-\infty}^{\infty} \rho_X(x)\mathrm{d}x = \int_{-\infty}^{\infty} \rho_Y(y)\mathrm{d}y = 1$. Let the product $\rho_X\rho_Y$ be normalised to secure probability conservation in $\mathbb{R}^2$ space:

$$P((X, Y) \in \mathbb{R}^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_X(x)\rho_Y(y)\mathrm{d}x\mathrm{d}y = 1. \quad (4)$$

All the points satisfying relation $\eta = \frac{x}{y}$ or alternatively $x = \eta y$ are localised on straight line passing through the axes origin with zero offset and with slope defined by knot efficiency. Similarly, points satisfying relation $\eta + \mathrm{d}\eta = \frac{x}{y}$ or alternatively $x = (\eta + \mathrm{d}\eta)y$ lie on a slightly more sloped straight line passing through the origin of axes [9]. Region $\mathrm{d}\Omega(\eta)$ between the two lines is the set of all the points with property $\eta < \frac{x}{y} < \eta + \mathrm{d}\eta$ (see Figure 1).

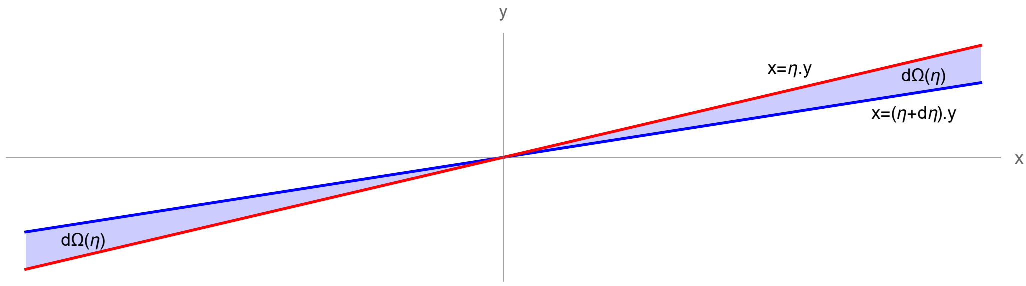

Figure 1. Region $\mathrm{d}\Omega(\eta)$ is the set of points between the two lines given by equations $\eta = \frac{x}{y}$ (red line) and $\eta + \mathrm{d}\eta = \frac{x}{y}$ (blue line).

Definition 2. *The region $\mathrm{d}\Omega(\eta)$ is the set of all the points $(x, y) \in \mathbb{R}^2$ bounded between lines $x = \eta y$ on the one side and $x = (\eta + \mathrm{d}\eta)y$ on the other (see Figure 1). Every point of this set refers to the knot efficiency from interval $(\eta, \eta + \mathrm{d}\eta)$, or in other words:*

$$\forall (x, y) \in [\mathrm{d}\Omega(\eta) \subset \mathbb{R}^2] : \eta < \frac{x}{y} < \eta + \mathrm{d}\eta. \tag{5}$$

Note that the red and blue line switch their relative positions as they cross the origin. Therefore, precise formulation of region $\mathrm{d}\Omega(\eta)$ requires the following set of inequalities:

$$\mathrm{d}\Omega(\eta) = \{(x, y) \in \mathbb{R}^2 | ([\eta y < x < (\eta + \mathrm{d}\eta)y] \wedge y > 0) \vee ([\eta y > x > (\eta + \mathrm{d}\eta)y] \wedge y < 0)\}. \tag{6}$$

At this point, all the preliminary issues have been addressed and we can introduce the knot efficiency PDF.

Definition 3. *Let $\eta = \frac{X}{Y}$ be a knot efficiency and $\mathrm{d}\eta \to 0^+$ be an infinitesimally small differential of the knot efficiency. Furthermore, let $\psi : \mathbb{R} \to [0, \infty)$ be a continuous, non-negative, Riemann-integrable function.*

Then we say that ψ is the knot efficiency PDF if following relations are satisfied:

- $P(\eta \le \frac{X}{Y} < \eta + \mathrm{d}\eta) = \int_{\eta}^{\eta+\mathrm{d}\eta} \psi(t)\mathrm{d}t = \psi(\eta)\mathrm{d}\eta;$
- $\int_{-\infty}^{\infty} \psi(\eta)\mathrm{d}\eta = 1.$

Definition 3 makes a connection between the knot efficiency and breaking strength:

$$\iint_{\mathrm{d}\Omega(\eta)} \rho_X(x)\rho_Y(y)\mathrm{d}x\mathrm{d}y = \psi(\eta)\mathrm{d}\eta. \tag{7}$$

Graphical representation of the knot efficiency PDF is shown in the Figure 2. The value of $\psi(\eta)$ is equal to the volume between the joint PDF ρ and the particular region $\mathrm{d}\Omega(\eta)$. The left side of this equation can be expanded with respect to the Definition 2:

$$\iint_{\mathrm{d}\Omega(\eta)} \rho_X(x)\rho_Y(y)\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{0} \rho_Y(y) \int_{(\eta+\mathrm{d}\eta)y}^{\eta y} \rho_X(x)\mathrm{d}x\mathrm{d}y + \int_{0}^{\infty} \rho_Y(y) \int_{\eta y}^{(\eta+\mathrm{d}\eta)y} \rho_X(x)\mathrm{d}x\mathrm{d}y \tag{8}$$

Attention should be paid to the pair of integrals $\int_{\eta y}^{(\eta+\mathrm{d}\eta)y} \rho_X(x)\mathrm{d}x$ and $\int_{(\eta+\mathrm{d}\eta)y}^{\eta y} \rho_X(x)\mathrm{d}x$, respectively. Function $\rho_X(x)$ is continuous, so there is room for significant simplification assuming the existence of indefinite integral $\int \rho_X(x)\mathrm{d}x = R(x) + C$:

$$\int_{\eta y}^{(\eta+\mathrm{d}\eta)y} \rho_X(x)\mathrm{d}x = R([\eta + \mathrm{d}\eta]y) - R(\eta y) = \frac{R(\eta y + \mathrm{d}\eta y) - R(\eta y)}{y\mathrm{d}\eta} y\mathrm{d}\eta \overset{\mathrm{d}\eta \to 0^+}{=} y\rho_X(y\eta)\mathrm{d}\eta. \tag{9}$$

Keep in mind that this holds true only for $y > 0$. The second integral can be simplified for $y < 0$ in a similar manner:

$$\int_{(\eta+\mathrm{d}\eta)y}^{\eta y} \rho_X(x)\mathrm{d}x = R(\eta y) - R([\eta + \mathrm{d}\eta]y) = -\frac{R(\eta y + \mathrm{d}\eta y) - R(\eta y)}{y\mathrm{d}\eta} y\mathrm{d}\eta \overset{\mathrm{d}\eta \to 0^+}{=} (-y)\rho_X(y\eta)\mathrm{d}\eta. \tag{10}$$

Taking these results into account Equation (8) can be rewritten:

$$\iint_{\mathrm{d}\Omega(\eta)} \rho_X(x)\rho_Y(y)\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{0} (-y)\rho_X(y\eta)\rho_Y(y)\mathrm{d}\eta\mathrm{d}y + \int_{0}^{\infty} y\rho_X(y\eta)\rho_Y(y)\mathrm{d}\eta\mathrm{d}y =$$

$$= \int_{-\infty}^{\infty} |y|\rho_X(y\eta)\rho_Y(y)\mathrm{d}\eta\mathrm{d}y = \left(\int_{-\infty}^{\infty} |y|\rho_X(y\eta)\rho_Y(y)\mathrm{d}y\right)\mathrm{d}\eta = \psi(\eta)\mathrm{d}\eta \quad (11)$$

Theorem 1. *Let ρ_X be the PDF of knotted rope breaking strength and ρ_Y be the straight rope breaking strength PDF. Both of them are continuous and independently normalised, and their joint bivariate distribution $\rho(x, y) = \rho_X(x)\rho_Y(y)$ is normalised on $\mathbb{R}^2$: $\iint_{\mathbb{R}^2} \rho_X(x)\rho_Y(y)\mathrm{d}x\mathrm{d}y = 1$. Finally, let η be the knot efficiency given by Definition 1.*

Then the relation between ρ_X, ρ_Y and probability density function of the knot efficiency ψ (see Definition 3) is given by convolution:

$$\psi(\eta) = \int_{-\infty}^{\infty} |y|\rho_X(y\eta)\rho_Y(y)\mathrm{d}y. \quad (12)$$

Proof of Theorem 1. Theorem 1 has been proven directly in text by Equations (8)–(11). □

Equation (12) presented in Theorem 1 can also be derived by different mathematical pathways and it has already been published in research papers [10–13].

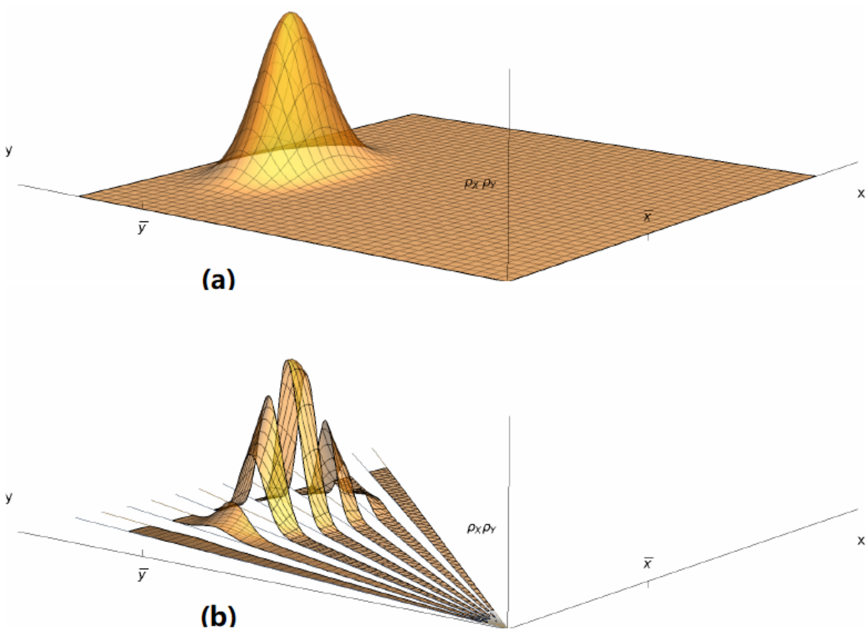

Figure 2. (**a**) Probability density function $\rho(x, y) = \rho_X(x)\rho_Y(y)$ on the upper figure. Volume under the PDF over the whole $\mathbb{R}^2$ space is equal to the probability of measuring any knot efficiency, i.e., $P = 1$. (**b**) On the lower picture, there are several slices of the PDF over the regions $\mathrm{d}\Omega_i$. Volume between the PDF and particular region is equal to the probability of measuring knot efficiency in range $[\eta_i, \eta_i + \mathrm{d}\eta]$.

2.2. *Probability Density Function of the Knot Efficiency for Normally Distributed Breaking Strength*

Theorem 1 gave us a blueprint how to calculate the knot efficiency PDF for arbitrarily distributed breaking strength of rope with or without a knot. However, it has been shown by [2] that ρ_X and ρ_Y of real ropes (knotted or straight) can be satisfactorily modelled by normal distribution with high level of accuracy. Another good reason why normal distribution should be used is the maximum information entropy principle [14–16] which favours the normal distribution (or its truncated version) if mean value and variance is given [17]. The central limit theorem is the last argument in favour of normal distribution. The more processes influence the breaking strength experiment within measurement, the more different dispersive factors are summed, and the more values it is averaged over. Thus, the more it tends to be normally distributed about the mean value, no mater what distribution it initially obeyed [8,18,19]. Therefore, it is worth trying to find a closed-form solution [20,21] to the convolution (12) for this special case.

Definition 4. *Let X be the static breaking strength of knotted rope. It will be considered as a continuous random variable with sample space $D_X = \mathbb{R}$ and probability density function:*

$$\rho_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(x-\bar{x})^2}{2\sigma^2}}. \tag{13}$$

Here $\bar{x} = E(X) = \int_{-\infty}^{\infty} x\rho_X(x)\mathrm{d}x$ is mean value and $\sigma^2 = Var(X) = \int_{-\infty}^{\infty}(x-\bar{x})^2\rho_X(x)\mathrm{d}x$ is variance of static breaking strength of knotted rope. Value $X \in [a,b)$ is measured with probability

$$P(a \le X < b) = \int_a^b \rho_X(x)\mathrm{d}x = \frac{1}{2}\left(\mathrm{erf}\left[\frac{b-\bar{x}}{\sqrt{2}\sigma}\right] - \mathrm{erf}\left[\frac{a-\bar{x}}{\sqrt{2}\sigma}\right]\right). \tag{14}$$

Definition 5. *Let Y be the static breaking strength of straight rope without any knot. It will be considered as a continuous random variable with sample space $D_Y = \mathbb{R}$ and probability density function:*

$$\rho_Y(y) = \frac{1}{\sqrt{2\pi}\varsigma} \mathrm{e}^{-\frac{(y-\bar{y})^2}{2\varsigma^2}}. \tag{15}$$

Here $\bar{y} = E(Y) = \int_{-\infty}^{\infty} y\rho_Y(y)\mathrm{d}y$ is mean value and $\varsigma^2 = Var(Y) = \int_{-\infty}^{\infty}(y-\bar{y})^2\rho_Y(y)\mathrm{d}y$ is variance of static breaking strength of straight rope. Value $Y \in [c,d)$ is measured with probability

$$P(c \le Y < d) = \int_c^d \rho_Y(y)\mathrm{d}y = \frac{1}{2}\left(\mathrm{erf}\left[\frac{d-\bar{y}}{\sqrt{2}\varsigma}\right] - \mathrm{erf}\left[\frac{c-\bar{y}}{\sqrt{2}\varsigma}\right]\right). \tag{16}$$

Whenever it will be necessary to emphasize that breaking strength is normally distributed, notation $X \sim \mathcal{N}(\bar{x},\sigma^2)$ and $Y \sim \mathcal{N}(\bar{y},\varsigma^2)$ will be used in the following text.

Quantities X and Y are independent so Theorem 1 leads us towards the convolution problem:

$$\psi(\eta) = \frac{1}{2\pi\sigma\varsigma}\int_{-\infty}^{\infty} |y|\, \mathrm{e}^{-\frac{1}{2}\left[\left(\frac{y\eta-\bar{x}}{\sigma}\right)^2 + \left(\frac{y-\bar{y}}{\varsigma}\right)^2\right]}\mathrm{d}y. \tag{17}$$

After applying a huge variety of algebraic transformations and parametric integration techniques (for details see the Proof of Theorem 3), integral (17) boils down to relatively complex, yet closed-form formula. Before we introduce it by Theorem 3, let us define new parameters p, q, r necessary to keep mathematical formulae as simple as possible.

Definition 6. *Let us introduce a triplet of new parameters p, q, r using quartet of known parameters $\bar{x}, \bar{y}, \sigma, \varsigma \in \mathbb{R}^+$ by following equations:*

$$p = \frac{\bar{x}}{\sqrt{2}\sigma} \qquad q = \frac{\bar{y}}{\sqrt{2}\varsigma} \qquad r = \frac{\bar{x}}{\bar{y}} \tag{18}$$

Let $\vec{v} = (p,q,r)$ be a vector in the space $\Pi \subset \mathbb{R}^3$ and $\Xi \subset \Pi$ be a sector in this space restricted by set of relations:

$$\Xi = \{(p,q,r) \in \Pi | 1 < p, 5 \le q, 0 < r \le 1\}. \tag{19}$$

Theorem 2. *Parameters p, q, r introduced by Definition 6 have following practical properties:*

1. *p is related to the probability of breaking strength X being a negative (nonphysical) number: $P(X < 0) = \frac{1-\text{erf}(p)}{2}$;*
2. *q is related to probability of breaking strength Y being a negative (nonphysical) number: $P(Y < 0) = \frac{1-\text{erf}(q)}{2}$;*
3. *r is ratio of expected values $r = \frac{E(X)}{E(Y)}$.*

Proof of Theorem 2. Let us start with statement 1. Probability $P(X < 0)$ is given by integral $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{0} \mathrm{e}^{-\frac{(x-\bar{x})^2}{2\sigma^2}}\mathrm{d}x$. After substitution $t = -\frac{x-\bar{x}}{\sqrt{2}\sigma}$ it transforms to integral

$$P(X < 0) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{0} \mathrm{e}^{-\frac{(x-\bar{x})^2}{2\sigma^2}}\mathrm{d}x = -\frac{1}{\sqrt{\pi}}\int_{\infty}^{\frac{\bar{x}}{\sqrt{2}\sigma}} \mathrm{e}^{-t^2}\mathrm{d}t =$$

$$= \frac{1}{\sqrt{\pi}}\int_{\frac{\bar{x}}{\sqrt{2}\sigma}}^{\infty} \mathrm{e}^{-t^2}\mathrm{d}t = \frac{1}{2}\left(\frac{2}{\sqrt{\pi}}\int_{0}^{\infty} \mathrm{e}^{-t^2}\mathrm{d}t - \frac{2}{\sqrt{\pi}}\int_{0}^{\frac{\bar{x}}{\sqrt{2}\sigma}} \mathrm{e}^{-t^2}\mathrm{d}t\right). \quad (20)$$

Recall that $\text{erf}(z)$ is so called *error function* defined as $\text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-t^2}\mathrm{d}t$, asymptotically converging to $\lim_{z\to\pm\infty}\text{erf}(z) = \pm 1$ [22]. Taking definition of error function into account, we may conclude:

$$P(X < 0) = \frac{1}{2}\left(\lim_{t\to\infty}\text{erf}(t) - \text{erf}\left[\frac{\bar{x}}{\sqrt{2}\sigma}\right]\right) = \frac{1-\text{erf}(p)}{2}. \quad (21)$$

Statement 2 would be proven by the same reasoning. Proof of statement 3 is straight forward and doesn't require our attention. □

Assumption 1. *Parameters p, q, r defined by Definition 6 and calculated from real data has following property:*

$$\vec{v} = (p, q, r) \in \Xi. \quad (22)$$

Assumption 1 is based on robust experimental evidence [1,2]. It can be taken as a *bomb-proof* fact that every real knot breaks in average under lower tension than straight rope. Thus, we are allowed to assume relation $0 < \bar{x} \le \bar{y}$, which implies $0 < \frac{\bar{x}}{\bar{y}} \le 1$ or $r \in (0,1]$. Parameters p, q, r for real knots have such values that probability $P(X < 0)$ and $P(Y < 0)$ is nonzero, but negligibly small [2]. Several examples of how small these numbers may actually be are calculated in Table 1.

Table 1. Several tabulated probabilities of negative breaking strength as a function of parameter q calculated using formula $P(Y < 0) = \frac{1-\text{erf}(q)}{2}$.

q	1	2	3	4	5
$\mathbf{P(Y < 0)}$	$7.86 \cdot 10^{-2}$	$2.34 \cdot 10^{-3}$	$1.10 \cdot 10^{-5}$	$7.71 \cdot 10^{-9}$	$7.69 \cdot 10^{-13}$

Now we are done with all the preliminaries. Let us make a step forward and introduce the knot efficiency PDF for normally distributed breaking strengths.

Theorem 3. *Let variable $X \sim \mathcal{N}(\bar{x}, \sigma^2)$ and $Y \sim \mathcal{N}(\bar{y}, \varsigma^2)$ be normally distributed with respect to Definition 4 and 5. Let us define parameters p, q, r the same way as in Definition 6.*

Let $s : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function defined by equation:

$$s(t) = q\frac{1+\frac{p^2}{q^2}t}{\sqrt{1+\frac{p^2}{q^2}t^2}}. \tag{23}$$

Than the knot efficiency $\eta = \frac{X}{Y}$ is distributed by PDF $\psi : \mathbb{R} \to \mathbb{R}^+$:

$$\psi(\eta) = \frac{1}{\pi}\frac{p}{q}e^{-q^2(1+\frac{p^2}{q^2})}\frac{1+\sqrt{\pi}s(\frac{\eta}{r})e^{s^2(\frac{\eta}{r})}\mathrm{erf}[s(\frac{\eta}{r})]}{1+\frac{p^2}{q^2}(\frac{\eta}{r})^2}\frac{1}{r}. \tag{24}$$

Proof of Theorem 3. Proof is quite simple, mostly subject to basic algebra, but it demands time and effort. Therefore, we sketch only the key ideas and leave the remaining calculations to the reader. The central problem is to find integral $\psi(\eta) = \frac{1}{2\pi\sigma\varsigma}\int_{-\infty}^{\infty}|y|\,e^{-\frac{1}{2}\left[\left(\frac{y\eta-\bar{x}}{\sigma}\right)^2+\left(\frac{y-\bar{y}}{\varsigma}\right)^2\right]}dy$. Let's expand the exponent: $-\frac{1}{2}\left[\left(\frac{y\eta-\bar{x}}{\sigma}\right)^2+\left(\frac{y-\bar{y}}{\varsigma}\right)^2\right] = \kappa y^2+\lambda y+\nu$. Coefficients κ, λ, ν have been introduced for the sake of simplicity:

$$\kappa = -\frac{q^2}{\bar{y}^2}\left(1+\frac{p^2}{q^2}\left[\frac{\eta}{r}\right]^2\right) \leq 0 \quad ; \quad \lambda = 2\frac{q^2}{\bar{y}}\left(1+\frac{p^2}{q^2}\left[\frac{\eta}{r}\right]\right) \geq 0 \quad ; \quad \nu = -(p^2+q^2) \leq 0.$$

The exponent can be rewritten by completing the square:

$$\kappa y^2+\lambda y+\nu = \kappa\left(y+\frac{\lambda}{2\kappa}\right)^2+\nu-\frac{\lambda^2}{4\kappa} = -|\kappa|\left(y-\frac{\lambda}{2|\kappa|}\right)^2-|\nu|+\frac{\lambda^2}{4|\kappa|}, \tag{25}$$

hence we are solving much simpler integral:

$$\int_{-\infty}^{\infty}|y|\,e^{\kappa y^2+\lambda y+\nu}dy = e^{-|\nu|+\frac{\lambda^2}{4|\kappa|}}\int_{-\infty}^{\infty}|y|\,e^{-|\kappa|\left(y-\frac{\lambda}{2|\kappa|}\right)^2}dy \tag{26}$$

Recall that for $\forall k, t_0 > 0$ the solution to the parametric integral $\int_{-\infty}^{\infty}|t|e^{-k(t-t_0)^2}dt$ is:

$$\int_{-\infty}^{\infty}|t|e^{-k(t-t_0)^2}dt = \frac{e^{-kt_0^2}}{k}\left[1+\sqrt{\pi}\sqrt{k}t_0e^{kt_0^2}\mathrm{erf}[\sqrt{k}t_0]\right]. \tag{27}$$

The integration is finished by substitution $k \to |\kappa|$ and $t_0 \to \frac{\lambda}{2|\kappa|}$. An ubiquitous term $\sqrt{k}t_0$ transforms to $\sqrt{k}t_0 \to \frac{\lambda^2}{4|\kappa|} = s(\frac{\eta}{r})$. The rest of the proof is straightforward, requiring mostly basic algebra resulting in formula: $\psi(\eta) = \frac{1}{\pi}\frac{p}{q}e^{-q^2(1+\frac{p^2}{q^2})}\frac{1+\sqrt{\pi}s(\frac{\eta}{r})e^{s^2(\frac{\eta}{r})}\mathrm{erf}[s(\frac{\eta}{r})]}{1+\frac{p^2}{q^2}(\frac{\eta}{r})^2}\frac{1}{r}$. □

Similar conclusions, but with different parametrisation or in different algebraic forms has already been published in papers dealing with ratio distributions, for example [12,23–28].

Several examples of real data knot efficiencies as well as ill-defined ones are sketched in Figure 3.

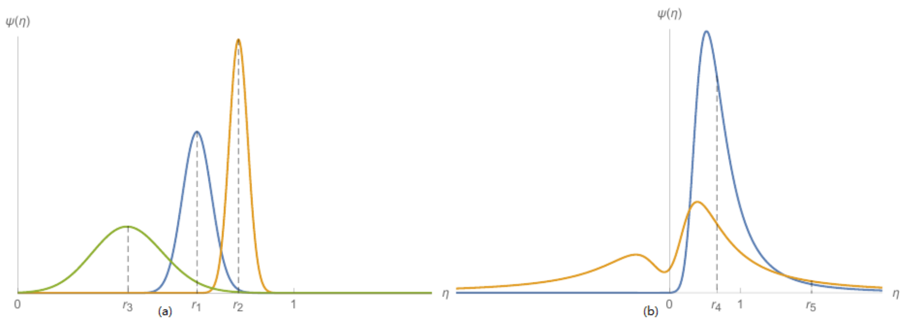

Figure 3. (**a**) Example of the knot efficiency PDFs for three different real life situations on the left side. PDF $\psi_1(\eta)$ depicted by blue line belongs to the standardly loaded figure eight loop knot tied on brand new kernmantle static rope. PDF $\psi_2(\eta)$ depicted by a yellow line belongs to the double fisherman's bend tied on brand new kernmantle static rope. PDF $\psi_3(\eta)$ depicted by the green line belongs to the cross-loaded figure eight loop tied on heterogenous, extremely worn static rope at the very end of its mission. The area under each curve equals 1. (**b**) An example of two knot efficiency PDFs with extraordinarily tuned parameters $(p, q, r) \notin \Xi$ that could never be realised in a real experiment.

2.3. Properties of Knot Efficiency PDF $\psi(\eta)$

1. Knotted rope breaking strength PDF is parameterised by two parameters $\bar{x}, \sigma$ and straight rope breaking strength PDF by another two parameters $\bar{y}, \varsigma$. However, knot efficiency PDF is fully parameterised only by three parameters p, q, r instead of four (see Theorem 3, Equation (24)). Other parameterisations employing another triplets of parameters are also possible.
2. Knot efficiency PDF is not a *symmetric function* under general reflection. Although it may seem symmetric upon first sight (see Figure 3a), it is important to stress that, depending on parameters p, q, r it may be noticeably tailed or even bimodal. This is especially apparent if $(p, q, r) \notin \Xi$ (see Figure 3b).
   In other words, for an arbitrary choice of parameters p, q, r there does not exist such a point η_0 that $\psi(\eta)$ would not change with respect to reflection over it:
$$\exists (p, q, r) \in \mathbb{R}^3,\ \forall \eta_0 \in \mathbb{R},\ \exists \eta \in \mathbb{R} : \psi(\eta_0 + \eta) \neq \psi(\eta_0 - \eta). \tag{28}$$
3. *Cumulative distribution function* of knot efficiency $\Psi(\eta)$.

   Definition 7. *Let $\psi(\eta) : \mathbb{R} \to \mathbb{R}^+$ be the knot efficiency PDF defined in Theorem 3 by Equation (24). Probability that knot efficiency η drawn from PDF $\psi(\eta)$ is lower than η_0 is given by cumulative distribution function $\Psi : \mathbb{R} \to [0, 1]$*
$$P(\eta < \eta_0) = \Psi(\eta_0) = \int_{-\infty}^{\eta_0} \psi(\eta) \mathrm{d}\eta. \tag{29}$$
   Function $\Psi(\eta)$ is continuous, non-decreasing (because $\forall \eta \in \mathbb{R} : \psi(\eta) \geq 0$) and satisfies properties $\lim_{\eta \to -\infty} \Psi(\eta) = 0$ and $\lim_{\eta \to \infty} \Psi(\eta) = 1$.
   It can be used to calculate probability: $P(a \leq \eta < b) = \int_a^b \psi(\eta)\mathrm{d}\eta = \Psi(b) - \Psi(a)$. Unfortunately, it is not possible to express $\Psi(\eta)$ as a finite combination of elementary functions because integral $\int_a^b \psi(\eta)\mathrm{d}\eta$ does not have a closed-form solution. This is somewhat inconvenient because $\Psi(\eta)$ is essential to calculate probabilities, tolerance intervals, and quantiles. To evaluate any practical outcome, it is required to involve numerical methods. Another possibility is to use Hinkley's formula [12] using so called *Bivariate normal distribution function* $L(x_1, x_2, x_3)$ numerically tabulated by the National Bureau of Standards [29] or some of its approximations [30,31].
4. An interesting feature of Equation (24) is that it entirely depends on argument $\frac{\eta}{r}$. Ratio $\frac{p}{q}$ is also omnipresent across the whole body of ψ. Let us call the quantity $\frac{\eta}{r}$ the *relative knot efficiency* and reserve symbol μ to label it. Substitution $\frac{\eta}{r} \to \mu$ is directly

coupled with related differential transformation $\frac{1}{r}\mathrm{d}\eta = \mathrm{d}\mu$. Probability must remain invariant under such a transformation, so the following equation holds true:

$$P(a \leq \eta < b) = \int_a^b \psi(\eta)\mathrm{d}\eta = \int_a^b \frac{1}{\pi}\frac{p}{q}\mathrm{e}^{-q^2(1+\frac{p^2}{q^2})}\frac{1+\sqrt{\pi}s(\frac{\eta}{r})\mathrm{e}^{s^2(\frac{\eta}{r})}\mathrm{erf}[s(\frac{\eta}{r})]}{1+\frac{p^2}{q^2}(\frac{\eta}{r})^2}\underbrace{\frac{1}{r}\mathrm{d}\eta}_{\mathrm{d}\mu} =$$

$$= \int_{\frac{a}{r}}^{\frac{b}{r}} \underbrace{\frac{1}{\pi}w\mathrm{e}^{-q^2(1+w^2)}\frac{1+\sqrt{\pi}s(\mu)\mathrm{e}^{s^2(\mu)}\mathrm{erf}[s(\mu)]}{1+w^2\mu^2}}_{\phi(\mu)}\mathrm{d}\mu = \Phi(\frac{b}{r}) - \Phi(\frac{a}{r}) \quad (30)$$

Several new elements have been introduced to simplify the expressions so let us define them properly:

Definition 8. *Let us define a new dimensionless quantity* $\mu = \frac{\eta}{r}$ *and name it relative knot efficiency.*

- *Let* $w \stackrel{def}{=} \frac{p}{q} = \frac{\bar{x}\varsigma}{\bar{y}\sigma}$ *be a new parameter. If* $\vec{v} \in \Xi$*, then parameter* w *falls into range* $w \in \mathbb{R}^+$*;*
- *PDF of relative knot efficiency is the function* $\phi(\mu)$*:*

$$\phi(\mu) \stackrel{def}{=} \frac{1}{\pi}w\mathrm{e}^{-q^2(1+w^2)}\frac{1+\sqrt{\pi}s(\mu)\mathrm{e}^{s^2(\mu)}\mathrm{erf}[s(\mu)]}{1+w^2\mu^2}. \quad (31)$$

- *cumulative distribution function of relative knot efficiency is the function* $\Phi(\mu)$*:*

$$\Phi(\mu) \stackrel{def}{=} \int_{-\infty}^{\mu} \phi(t)\mathrm{d}t; \quad (32)$$

Note that relative knot efficiency PDF ϕ is completely parameterised only by two parameters w, q.

5. The relation between Ψ and Φ is given by probability conservation principle. Equality of the following integrals is guaranteed $\int_{-\infty}^{\eta} \psi(t)\mathrm{d}t = \int_{-\infty}^{\frac{\eta}{r}} \phi(t)\mathrm{d}t$, or:

$$\Psi(\eta) = \Phi(\frac{\eta}{r}); \quad (33)$$

6. Probability that knot efficiency is a non-physical negative number is invariant of r and equals: $P(\eta < 0) = \Psi(0) = \Phi(0) = \int_{-\infty}^{0} \phi(t)\mathrm{d}t$. In order to become familiar with real knots values of $\Psi(0)$, they have been calculated for selected combinations of parameters w, q by means of numerical integration (see Table 2). It is clear that $P(\eta < 0)$ of real knots is negligibly small.

Table 2. Probability $P(\eta < 0) = \Psi(0)$ that non-physical negative knot efficiency will be measured, tabulated for selected combinations of parameters w, q.

	w = 0.2	w = 0.3	w = 0.4	w = 0.5	w = 0.6	w = 0.7	w = 0.8	w = 0.9	w = 1
q = 1	$3.9 \cdot 10^{-1}$	$3.4 \cdot 10^{-1}$	$2.9 \cdot 10^{-1}$	$2.4 \cdot 10^{-1}$	$2.0 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$	$1.0 \cdot 10^{-1}$	$7.9 \cdot 10^{-2}$
q = 2	$2.9 \cdot 10^{-1}$	$2.0 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$	$7.9 \cdot 10^{-2}$	$4.5 \cdot 10^{-2}$	$2.4 \cdot 10^{-2}$	$1.2 \cdot 10^{-2}$	$5.5 \cdot 10^{-3}$	$2.3 \cdot 10^{-3}$
q = 3	$2.0 \cdot 10^{-1}$	$1.0 \cdot 10^{-1}$	$4.5 \cdot 10^{-2}$	$1.7 \cdot 10^{-2}$	$5.5 \cdot 10^{-3}$	$1.5 \cdot 10^{-3}$	$3.4 \cdot 10^{-4}$	$6.7 \cdot 10^{-5}$	$1.1 \cdot 10^{-5}$
q = 4	$1.3 \cdot 10^{-1}$	$4.5 \cdot 10^{-2}$	$1.2 \cdot 10^{-2}$	$2.3 \cdot 10^{-3}$	$3.4 \cdot 10^{-4}$	$3.8 \cdot 10^{-5}$	$3.0 \cdot 10^{-6}$	$1.8 \cdot 10^{-7}$	$7.7 \cdot 10^{-9}$
q = 5	$8.1 \cdot 10^{-2}$	$1.7 \cdot 10^{-2}$	$2.3 \cdot 10^{-3}$	$2.0 \cdot 10^{-4}$	$1.1 \cdot 10^{-5}$	$3.7 \cdot 10^{-7}$	$7.7 \cdot 10^{-9}$	$9.8 \cdot 10^{-11}$	$7.7 \cdot 10^{-13}$
q = 6	$4.7 \cdot 10^{-2}$	$5.5 \cdot 10^{-3}$	$3.4 \cdot 10^{-4}$	$1.1 \cdot 10^{-5}$	$1.8 \cdot 10^{-7}$	$1.4 \cdot 10^{-9}$	$5.7 \cdot 10^{-12}$	$1.1 \cdot 10^{-14}$	$1.1 \cdot 10^{-17}$
q = 7	$2.6 \cdot 10^{-2}$	$1.5 \cdot 10^{-3}$	$3.8 \cdot 10^{-5}$	$3.7 \cdot 10^{-7}$	$1.4 \cdot 10^{-9}$	$2.1 \cdot 10^{-12}$	$1.2 \cdot 10^{-15}$	$2.6 \cdot 10^{-19}$	$2.1 \cdot 10^{-23}$
q = 8	$1.4 \cdot 10^{-2}$	$3.6 \cdot 10^{-4}$	$3.0 \cdot 10^{-6}$	$7.7 \cdot 10^{-9}$	$5.7 \cdot 10^{-12}$	$1.2 \cdot 10^{-15}$	$7.1 \cdot 10^{-20}$	$1.2 \cdot 10^{-24}$	$5.6 \cdot 10^{-30}$
q = 9	$7.8 \cdot 10^{-3}$	$7.8 \cdot 10^{-5}$	$1.9 \cdot 10^{-7}$	$9.9 \cdot 10^{-11}$	$1.1 \cdot 10^{-14}$	$2.6 \cdot 10^{-19}$	$1.2 \cdot 10^{-24}$	$1.1 \cdot 10^{-30}$	$2.1 \cdot 10^{-37}$
q = 10	$4.7 \cdot 10^{-3}$	$2.2 \cdot 10^{-5}$	$1.5 \cdot 10^{-8}$	$1.5 \cdot 10^{-12}$	$2.2 \cdot 10^{-17}$	$4.2 \cdot 10^{-23}$	$1.1 \cdot 10^{-29}$	$4.1 \cdot 10^{-37}$	$2.1 \cdot 10^{-45}$

7. Let $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$ be non-negative integers. The probability that knot efficiency falls into range $[mr, nr)$ does not depend on r and equals:

$$P(mr \leq \eta < nr) = \Phi(n) - \Phi(m) = \int_m^n \phi(t)\mathrm{d}t; \tag{34}$$

8. Knot efficiency PDF (see Equation (24)) does not have a finite mean or variance. It is a relatively easy task to prove that integrals $\int_{-\infty}^{\infty} \eta\psi(\eta)\mathrm{d}\eta$ and $\int_{-\infty}^{\infty} \eta^2\psi(\eta)\mathrm{d}\eta$ are divergent. Therefore, expected value $E(\eta)$ is not finite (i.e., does not exist) and the same goes for variance $Var(\eta)$ [32,33].
Moreover, real knots never break at negative breaking strength, and a knotted rope breaking higher than a straight one is extremely rare (to our knowledge it has never been reported yet). Function ψ is continuous, non-negative, and bounded for $\forall \eta \in [0,1]$. Therefore, it is well reasoned to calculate certainly finite mean $\bar{\eta}_{[0,1]}$ and variance $\sigma^2_{[0,1]}$ of truncated knot efficiency PDF, restricted on interval $\eta \in [0,1]$.

Definition 9. *Let ψ be the knot efficiency PDF and Ψ be the knot efficiency CDF introduced by Definition 7. Let ϕ be the solid approximation of relative knot efficiency PDF and Φ be the solid approximation of relative knot efficiency CDF introduced by Definition 8. Then, the expected value of knot efficiency $\bar{\eta}_{[0,1]}$ on interval $\eta \in [0,1]$ is defined by equation:*

$$\bar{\eta}_{[0,1]} = \frac{\int_0^1 \eta\psi(\eta)\mathrm{d}\eta}{\Psi(1) - \Psi(0)} = r\frac{\int_0^{1/r} \mu\phi(\mu)\mathrm{d}\mu}{\Phi(1/r) - \Phi(0)}. \tag{35}$$

Analogically, variance of knot efficiency $\sigma^2_{\eta[0,1]}$ on interval $\eta \in [0,1]$ is defined by equation:

$$\sigma^2_{\eta[0,1]} = \frac{\int_0^1 (\eta - \bar{\eta}_{[0,1]})^2\psi(\eta)\mathrm{d}\eta}{\Psi(1) - \Psi(0)} = r^2\frac{\int_0^{1/r} \left(\mu - \frac{\bar{\eta}_{[0,1]}}{r}\right)^2 \phi(\mu)\mathrm{d}\mu}{\Phi(1/r) - \Phi(0)}. \tag{36}$$

To our knowledge, both $\bar{\eta}_{[0,1]}$ and $\sigma^2_{\eta[0,1]}$ have to be evaluated by numerical integration techniques.

9. In general, neither mode nor median shall be identified with ratio $r = \frac{\bar{x}}{\bar{y}}$ as one may intuitively suppose. The median knot efficiency $\tilde{\eta}$ can be evaluated only by means of numerical integration by solving integral equation $\int_{-\infty}^{\tilde{\eta}} \psi(\eta)\mathrm{d}\eta = \int_{\tilde{\eta}}^{\infty} \psi(\eta)\mathrm{d}\eta = \frac{1}{2}$ or

$$\Psi(\tilde{\eta}) = 1 - \Psi(\tilde{\eta}) = \frac{1}{2}. \tag{37}$$

It will be shown later in the text (see Theorem 5), that median $\tilde{\eta}$ and ratio $r = \frac{\bar{x}}{\bar{y}}$ of knot efficiency PDFs can be considered almost equal.

10. PDF $\psi(\eta)$ is in general bimodal [24], see Figure 3b. This means that it has two local maxima instead of one. According to numerical studies presented in [32], PDF is certainly bimodal if $\sqrt{2}p > 2.256058904\cdots \equiv C$. If $\sqrt{2}p < C$ then the PDF may be either bimodal or unimodal, depending on the exact position of vector $\vec{v}$ in the space of the parameters. There is a region $K \subset \mathbb{R}^+ \times \mathbb{R}^+$ separated from the rest of space by a simple curve $\Gamma = \partial K$ such that for $\forall (p,q) \in, K$ is ψ unimodal. A good rule of thumb to make an accurate decision about PDF modality is to check the following two conditions:
    - $0 < p < 1.6$;
    - $5.7 < q$.

    If both of them are satisfied simultaneously then knot efficiency PDF is almost certainly unimodal.
    However, one should keep in mind that the left modes of bimodal functions may be ignored in practical applications, as they are likely to occupy only a tiny fraction of the total area under the full density function, typically 10^{-6} to 10^{-10} or less [32].
    The mode of knot efficiency η_m is the most likely value to be drawn from knot efficiency population. If the population PDF is continuously differentiable then the mode is a solution to the pair of conditions:

$$\frac{\mathrm{d}\psi}{\mathrm{d}\eta}(\eta_m) = 0 \qquad , \qquad \frac{\mathrm{d}^2\psi}{\mathrm{d}\eta^2}(\eta_m) < 0. \tag{38}$$

11. About $\mathrm{erf}\left[\frac{1}{\sqrt{2}}\right] \approx 68\%$ of values drawn from a normal distribution are within one standard deviation away from the mean. This is a so-called 1σ tolerance interval well known in statistics, or $[\bar{x}-\sigma, \bar{x}+\sigma]$. About $\mathrm{erf}\left[\frac{2}{\sqrt{2}}\right] \approx 95\%$ of the values lie within two standard deviations and about $\mathrm{erf}\left[\frac{3}{\sqrt{2}}\right] \approx 99.7\%$ within three standard deviations. For practical purposes, similar thresholds are also needed to be defined for knot efficiency PDF.
    - Knot efficiency analogy to 1σ tolerance interval (see Figure 4) is $[\eta_{1L}, \eta_{1H}]$. Thresholds η_{1L}, η_{1H} have to meet pair of conditions: $\Psi(\eta_{1L}) = 1 - \Psi(\eta_{1H}) = \frac{1}{2}\left(1 - \mathrm{erf}\left[\frac{1}{\sqrt{2}}\right]\right)$;
    - In general, knot efficiency analogy to $n\sigma$ interval for $\forall n \in \mathbb{N}$ is $[\eta_{nL}, \eta_{nH}]$. The following pair of conditions have to be fulfilled:

$$\Psi(\eta_{nL}) = 1 - \Psi(\eta_{nH}) = \frac{1}{2}\left(1 - \mathrm{erf}\left[\frac{n}{\sqrt{2}}\right]\right). \tag{39}$$

    Finding thresholds η_{nL}, η_{nH} is subject to the numerical integration and depends on triplet of parameters $\{p,q,r\}$ or $\{w,q,r\}$.

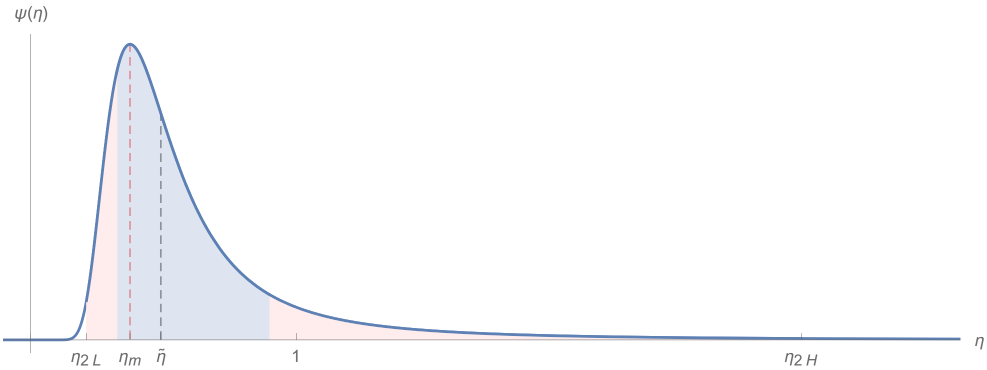

Figure 4. An example of heavily tailed knot efficiency PDF. Important points and intervals are highlighted—modus (red dashed line), median (black dashed line), analogy to 1σ interval $[\eta_{1L}, \eta_{1H}]$ is highlighted by blue shading, remaining interval to 2σ analogy is highlighted by pink shading.

2.4. Solid Approximation to the Knot Efficiency PDF

Knot efficiency probability density function $\psi(\eta)$ given by Equation (24) is much too complex; a closed-form solution to its Riemann integral does not exist, and many related statistical properties are subject to numerical methods. These unwanted properties apparently narrow the field of application for a standard user without expertise in numerical mathematics. However, in many cases, it can be considerably simplified without significant loss of accuracy. Depending on parameters p, q, r, function $\psi(\eta)$ is more or less tailed. The narrower and more symmetrical it is, the stronger simplification actions can be taken, resulting in improved user manipulation comfort. Fortunately, real knots populations are distributed with a considerable degree of mirror symmetry, and similarity with normal distribution is usually quite high.

Presented approximation has been named solid. Loosely speaking, this is because it remains highly accurate while offering significant simplification and closed-form integrability at the same time. The simplifying chain consists of following three steps:

- Parameters p, q, r can be in general set to an arbitrary value. However, in the case of real knots, wide range is ruled out by laws of physics; hence, they will never be observed in real measurement. It has already been outlined in Assumption 1 that the vector of parameters $\vec{v} = (p, q, r)$ always points into region $\vec{v} \in \Xi$ for real data. Regarding the mentioned restrictions, function $s(\frac{\eta}{r})$ is bounded:

$$\forall \eta \in \mathbb{R}^+, \vec{v} \in \Xi : \min\{p, q\} \leq s(\frac{\eta}{r}) \leq \sqrt{p^2 + q^2}; \tag{40}$$

- Error function $\text{erf}(t)$ is monotonous increasing, as t increases asymptotically reaching $\lim_{t \to \infty} \text{erf}(t) = 1$. Hence, we are able to bound it:

$$\forall \eta \in \mathbb{R}^+, \vec{v} \in \Xi : \min\{\text{erf}(p), \text{erf}(q)\} \leq \text{erf}[s(\eta/r)] \leq \text{erf}\left[\sqrt{p^2 + q^2}\right] \approx 1. \tag{41}$$

  The lower bound of $s(\eta/r)$ is for $\forall \vec{v} \in \Xi$ so close to the upper one, that with accuracy far beyond all practical needs we can simply write: $\forall \eta \in [0, 1]$, $\vec{v} \in \Xi : \text{erf}[s(\eta/r)] \approx 1$;
- Term $\sqrt{\pi} s(\eta/r) \text{e}^{s^2(\frac{\eta}{r})}$ is many orders of magnitude higher than than 1. Therefore, we can suppose:

$$1 + \sqrt{\pi} s(\eta/r) \text{e}^{s^2(\frac{\eta}{r})} \approx \sqrt{\pi} s(\eta/r) \text{e}^{s^2(\frac{\eta}{r})}; \tag{42}$$

- Concluding the above-mentioned reasoning, we are able to say:

$$\forall \eta \in \mathbb{R}^+, \vec{v} \in \Xi : 1 + \sqrt{\pi} s(\eta/r) \text{e}^{s^2(\frac{\eta}{r})} \text{erf}[s(\eta/r)] \approx \sqrt{\pi} s(\eta/r) \text{e}^{s^2(\frac{\eta}{r})}, \tag{43}$$

and after some algebraic manipulations we can proceed to approximation

$$\psi(\eta) \approx \psi^{\ddagger}(\eta)\frac{p}{\sqrt{\pi}}\frac{1+\frac{p^2}{q^2}\frac{\eta}{r}}{\left[1+\frac{p^2}{q^2}\left(\frac{\eta}{r}\right)^2\right]^{\frac{3}{2}}}\mathrm{e}^{-\frac{p^2}{1+\frac{p^2}{q^2}\left(\frac{\eta}{r}\right)^2}\left(\frac{\eta}{r}-1\right)^2}\frac{1}{r}. \tag{44}$$

A similar equation can be written for the relative knot efficiency PDF:

$$\phi(\mu) \approx \phi^{\ddagger}(\mu) = \frac{wq}{\sqrt{\pi}}\frac{1+w^2\mu}{(1+w^2\mu^2)^{\frac{3}{2}}}\mathrm{e}^{-\frac{w^2q^2}{1+w^2\mu^2}(\mu-1)^2}. \tag{45}$$

The simplifying steps brought several benefits. The complexity of formulae has been lowered and it is finally possible to find closed-form solution to the Riemann-integral of $\psi^{\ddagger}$ and $\phi^{\ddagger}$. This property opens the door to precise normalisation and simple calculation of median, quantiles, and tolerance intervals.

Theorem 4. *Let $\phi^{\ddagger}:\mathbb{R}\to\mathbb{R}^{+}$ be continuous function defined by Equation (45). Then Riemann integral $\int_{-\infty}^{\mu}\phi^{\ddagger}(t)\mathrm{d}t$ has closed-form solution:*

$$\int_{-\infty}^{\mu}\phi^{\ddagger}(t)\mathrm{d}t = \frac{1}{2}\left(\mathrm{erf}\left[\frac{wq(\mu-1)}{\sqrt{1+w^2\mu^2}}\right]+\mathrm{erf}[q]\right). \tag{46}$$

Proof of Theorem 4. Function $\phi^{\ddagger}(\mu)$ is continuous for $\forall\mu\in\mathbb{R}$, therefore Riemann-integrable and we can write: $\int_{-\infty}^{\mu}\phi^{\ddagger}(t)\mathrm{d}t = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\mu} wq\frac{1+w^2t}{(1+w^2t^2)^{\frac{3}{2}}}\mathrm{e}^{-\frac{w^2q^2}{1+w^2t^2}(t-1)^2}\mathrm{d}t$. Now let us make a substitution $\frac{wq(t-1)}{\sqrt{1+w^2t^2}} = p$ coupled with differential transformation $wq\frac{1+w^2t}{(1+w^2t^2)^{\frac{3}{2}}}\mathrm{d}t = \mathrm{d}p$ and boundary transformations $-\infty\to -q$ and $\mu\to\frac{wq(\mu-1)}{\sqrt{1+w^2\mu^2}}$. Then, the original integral transforms to the much simpler one:

$$\begin{aligned}\int_{-\infty}^{\mu}\phi^{\ddagger}(t)\mathrm{d}t &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\mu} wq\frac{1+w^2t}{(1+w^2t^2)^{\frac{3}{2}}}\mathrm{e}^{-\frac{w^2q^2}{1+w^2t^2}(t-1)^2}\mathrm{d}t = \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_{-q}^{\frac{wq(\mu-1)}{\sqrt{1+w^2\mu^2}}}\mathrm{e}^{-p^2}\mathrm{d}p = \\ &= \frac{1}{2}[\mathrm{erf}(p)]_{-q}^{\frac{wq(\mu-1)}{\sqrt{1+w^2\mu^2}}} = \frac{1}{2}\left(\mathrm{erf}\left[\frac{wq(\mu-1)}{\sqrt{1+w^2\mu^2}}\right]+\mathrm{erf}[q]\right).\end{aligned}$$

□

Theorem 4 holds the key to finding normalisation factors. However, all the necessary conditions have been met to define *solid approximation* to the knot efficiency and relative knot efficiency, respectively.

Definition 10. *Let $\vec{v}\in\Xi$. Then, function $\psi^{\dagger}(\eta):\mathbb{R}\to\mathbb{R}^{+}$ is the solid approximation to the knot efficiency PDF given by equation:*

$$\psi^{\dagger}(\eta) = \frac{1}{\sqrt{\pi}}\frac{p}{\mathrm{erf}[q]}\frac{1+\frac{p^2}{q^2}\frac{\eta}{r}}{\left(1+\frac{p^2}{q^2}\frac{\eta^2}{r^2}\right)^{\frac{3}{2}}}\mathrm{e}^{-\frac{p^2\left(\frac{\eta}{r}-1\right)^2}{1+\frac{p^2}{q^2}\frac{\eta^2}{r^2}}}\frac{1}{r}. \tag{47}$$

Probability that knot efficiency η drawn from PDF $\psi^\dagger(\eta)$ is lower than η_0 is given by cumulative distribution function $\Psi^\dagger : \mathbb{R} \to [0,1]$

$$P(\eta < \eta_0) = \Psi^\dagger(\eta_0) = \int_{-\infty}^{\eta_0} \psi^\dagger(\eta)\mathrm{d}\eta = \frac{1}{2}\left(1 + \frac{\mathrm{erf}\left[\frac{p(\frac{\eta_0}{r}-1)}{\sqrt{1+\frac{p^2}{q^2}\frac{\eta_0^2}{r^2}}}\right]}{\mathrm{erf}[q]}\right). \tag{48}$$

Definition 11. *Let $\vec{v} \in \Xi$ and $w \in (0,1]$. Then function $\phi^\dagger(\mu) : \mathbb{R} \to \mathbb{R}^+$ is the solid approximation to the relative knot efficiency PDF given by equation:*

$$\phi^\dagger(\mu) = \frac{1}{\sqrt{\pi}}\frac{wq}{\mathrm{erf}[q]}\frac{1+w^2\mu}{(1+w^2\mu^2)^{\frac{3}{2}}}\mathrm{e}^{-\frac{w^2q^2}{1+w^2\mu^2}(\mu-1)^2}. \tag{49}$$

The probability that relative knot efficiency μ drawn from PDF $\phi^\dagger(\mu)$ is lower than μ_0 is given by cumulative distribution function $\Phi^\dagger : \mathbb{R} \to [0,1]$

$$P(\mu < \mu_0) = \Phi^\dagger(\mu_0) = \int_{-\infty}^{\mu_0} \phi^\dagger(\mu)\mathrm{d}\mu = \frac{1}{2}\left(1 + \frac{\mathrm{erf}\left[\frac{wq(\mu_0-1)}{\sqrt{1+w^2\mu_0^2}}\right]}{\mathrm{erf}[q]}\right). \tag{50}$$

Solid approximation addressed almost all the original drawbacks of PDFs $\psi(\eta)$ and $\phi(\eta)$. Significant improvement has been achieved in the following aspects:

1. Both PDF formulae $\psi^\dagger, \phi^\dagger$ are less complex than ψ, ϕ; hence, they are more suitable for practical calculations. Both of them are normalised, i.e., they obey integral relation $\int_{-\infty}^{\infty} \psi^\dagger(\eta)\mathrm{d}\eta = \int_{-\infty}^{\infty} \phi^\dagger(\eta)\mathrm{d}\eta = 1$. For any reasonable combination of parameters $\forall \vec{v} \in \Xi$, solid approximation is indistinguishable from ψ, ϕ.
2. Both CDFs $\Psi^\dagger$ and $\Phi^\dagger$ are closed-form formulae (see Definitions 10 and 11). Function $\Phi^\dagger$ is slightly simpler than $\Psi^\dagger$ so it is recommended to use it for calculations whenever possible. Relation $\Psi^\dagger(\eta) = \Phi^\dagger(\frac{\eta}{r})$ holds true.
3. Since $\Phi^\dagger$ is defined by explicit closed-form formula, median of solid approximation $\tilde{\eta}^\dagger$ can be simply calculated by the following equation: $\Phi^\dagger(\frac{\tilde{\eta}^\dagger}{r}) = \frac{1}{2}$.

Theorem 5 (Median $\tilde{\eta}^\dagger$)**.** *Let $\Psi^\dagger(\mu)$ be a solid approximation to relative knot efficiency CDF (see Definition 10). Then solid approximation to the median of knot efficiency $\tilde{\eta}^\dagger$ is:*

$$\tilde{\eta}^\dagger = \frac{\bar{x}}{\bar{y}}. \tag{51}$$

Proof of Theorem 5. Median $\tilde{\eta}^\dagger$ can be found from equation $\Psi^\dagger(\tilde{\eta}^\dagger) = \Phi^\dagger(\frac{\tilde{\eta}^\dagger}{r}) = \frac{1}{2}$. According to Definition 10, it is possible to rewrite the equation:

$$\mathrm{erf}\left[\frac{wq(\frac{\tilde{\eta}^\dagger}{r}-1)}{\sqrt{1+w^2\left(\frac{\tilde{\eta}^\dagger}{r}\right)^2}}\right] = 0. \tag{52}$$

The solution is $\tilde{\eta}^\dagger = r$. □

Position of the median $\tilde{\eta}^\dagger$ is shown in the Figure 5.

4. Solid approximation analogy to 1σ and 2σ tolerance interval can be found explicitly whenever condition $\vec{v} \in \Xi$ is met. Solutions to equations $\Psi^\dagger(\eta^\dagger_{nL}) = \frac{1}{2}\left(1 - \mathrm{erf}\left[\frac{n}{\sqrt{2}}\right]\right)$ and $1 - \Psi^\dagger(\eta^\dagger_{nH}) = \frac{1}{2}\left(1 - \mathrm{erf}\left[\frac{n}{\sqrt{2}}\right]\right)$ for $n \in \{1, 2\}$ are:
   - $1\sigma^\dagger$ interval: $[\eta^\dagger_{1L}, \eta^\dagger_{1H}] = \left[r\frac{1 - \frac{1}{\sqrt{2}wq}\sqrt{1+w^2-\frac{1}{2q^2}}}{1-\frac{1}{2q^2}}, r\frac{1 + \frac{1}{\sqrt{2}wq}\sqrt{1+w^2-\frac{1}{2q^2}}}{1-\frac{1}{2q^2}} \right]$
   - $2\sigma^\dagger$ interval: $[\eta^\dagger_{2L}, \eta^\dagger_{2H}] = \left[r\frac{1 - \frac{\sqrt{2}}{wq}\sqrt{1+w^2-\frac{2}{q^2}}}{1-\frac{2}{q^2}}, r\frac{1 + \frac{\sqrt{2}}{wq}\sqrt{1+w^2-\frac{2}{q^2}}}{1-\frac{2}{q^2}} \right]$

   Ranges of $1\sigma^\dagger$ and $2\sigma^\dagger$ intervals are shown in the Figure 5 by colour shading. Note that $1\sigma^\dagger$ and $2\sigma^\dagger$ formulae are valid approximations if $5 \leq q$. Otherwise, more complex closed-form formulae containing inverse function erf^{-1} can be easily derived.
5. It has already been stated that knot efficiency distributed by PDF $\psi(\eta)$ does not have a finite mean or variance. Solid approximation itself does not bring any progress on this issue because PDF $\psi^\dagger$ also leads to divergent integrals $\int_{-\infty}^{\infty} \eta\psi^\dagger(\eta)\mathrm{d}\eta$ or $\int_{-\infty}^{\infty} \eta^2\psi^\dagger(\eta)\mathrm{d}\eta$. Nonetheless, truncated mean and variance can be calculated on the interval $\eta \in [0, 1]$ based on the same principles as Definition 9.
   - Solid approximation to knot efficiency mean on the interval $\eta \in [0, 1]$:

$$\bar{\eta}^\dagger_{[0,1]} = \frac{\int_0^1 \eta\psi^\dagger(\eta)\mathrm{d}\eta}{\Psi^\dagger(1) - \Psi^\dagger(0)} = r\frac{\int_0^{1/r} \mu\phi^\dagger(\mu)\mathrm{d}\mu}{\Phi^\dagger(1/r) - \Phi^\dagger(0)}; \tag{53}$$

   - Solid approximation to knot efficiency variance on the interval $\eta \in [0, 1]$:

$$\sigma^{2\dagger}_{\eta[0,1]} = \frac{\int_0^1 (\eta - \bar{\eta}^\dagger_{[0,1]})^2\psi^\dagger(\eta)\mathrm{d}\eta}{\Psi^\dagger(1) - \Psi^\dagger(0)} = r^2\frac{\int_0^{1/r} \left(\mu - \frac{\bar{\eta}^\dagger_{[0,1]}}{r}\right)^2 \phi^\dagger(\mu)\mathrm{d}\mu}{\Phi^\dagger(1/r) - \Phi^\dagger(0)}. \tag{54}$$

   To our knowledge, both $\bar{\eta}^\dagger_{[0,1]}$ and $\sigma^{2\dagger}_{\eta[0,1]}$ have to be evaluated by numerical integration. Position of the truncated mean $\bar{\eta}^\dagger_{[0,1]}$ and standard deviation $\sigma^\dagger_{\eta[0,1]}$ is shown in the Figure 5 by dotted lines.

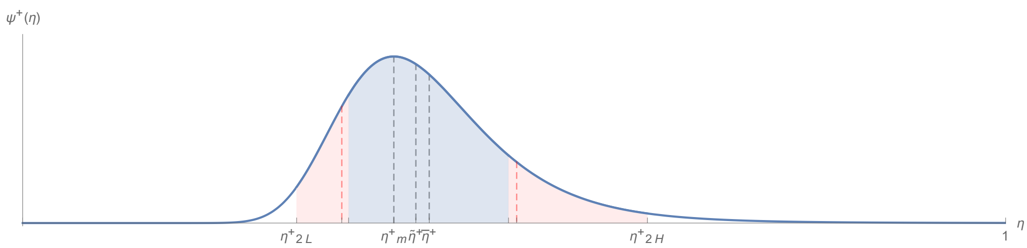

Figure 5. Comparison of knot efficiency PDF $\psi(\eta)$ and solid approximation of knot efficiency PDF $\psi^\dagger(\eta)$. In fact, the two functions overlap so precisely that they are indistinguishable. Important points and intervals are emphasized: modus $\eta^\dagger_m$, median $\tilde{\eta}^\dagger$, mean $\bar{\eta}^\dagger_{[0,1]}$, $1\sigma^\dagger$ interval is highlighted by blue shading, remaining interval to $2\sigma^\dagger$ is highlighted by pink shading. Red dashed lines cross the axis at points $\bar{\eta}^\dagger - \sqrt{\sigma^{2\dagger}_{\eta[0,1]}}$ and $\bar{\eta}^\dagger + \sqrt{\sigma^{2\dagger}_{\eta[0,1]}}$.

2.5. Solid Approximation Error Management

The very nature of every approximation is that some kinds of errors are inevitably introduced into calculations. Its estimation is crucial for management of the reliability of results. In the following text it will be shown that solid approximation $\psi^\dagger, \phi^\dagger, \Psi^\dagger, \Phi^\dagger$ accuracy is far beyond any practical needs, and that the benefit-error ratio is high enough to favour it against ψ, ϕ, Ψ, Φ. However, prior to making this step, it would be useful to propose a framework of auxiliary terms and functions in order to keep following theorems and proof formulae as simple as possible.

Definition 12. *Let us consider parameters p, q, r, w to be restricted by relations $p, q, w > 0$ and $0 < r \leq 1$. Let number $u \overset{def}{=} \min\{p, q\}$ be the smaller of the two values p, q. Let $\alpha \overset{def}{=} e^{-q^2(1+w^2)}$ and $\beta \overset{def}{=} [1 - \mathrm{erf}(q)\mathrm{erf}(u)]$ be two main error scaling factors. Functions $\phi^\dagger, \Phi^\dagger$ are taken from Definition 11. Then*

- *the* absolute PDF error $\delta^\dagger : \mathbb{R} \to \mathbb{R}^+$ *is defined by equation:*

$$\delta^\dagger(t) = \frac{\alpha}{\pi}\frac{w}{1 + w^2t^2} + \beta\phi^\dagger(t); \tag{55}$$

- *the* absolute CDF error $\Delta^\dagger : \mathbb{R} \to \mathbb{R}^+$ *is defined by equation:*

$$\Delta^\dagger(t) = \alpha\left[\frac{1}{2} + \frac{\arctan(wt)}{\pi}\right] + \beta\Phi^\dagger(t); \tag{56}$$

- *the* absolute mean error $\varepsilon^\dagger_{\bar{\eta}} \in \mathbb{R}^+$ *is a constant defined by equation:*

$$\varepsilon^\dagger_{\bar{\eta}} = \alpha\left[2 + \frac{\ln\left(1 + \frac{w^2}{r^2}\right)}{2\pi\frac{w}{r}}\right] + 3\beta; \tag{57}$$

- *the* absolute variance error $\varepsilon^\dagger_{\bar{\eta}} \in \mathbb{R}^+$ *is a constant defined by equation:*

$$\varepsilon^\dagger_{\sigma^2_{\bar{\eta}}} = \alpha\left[16 + \frac{(1 - \frac{r^2}{w^2})\arctan(\frac{w}{r}) + \frac{r}{w}\left(1 + 3\ln(1 + \frac{w^2}{r^2})\right)}{\pi}\right] + 23\beta. \tag{58}$$

Keep in mind that terms $\alpha = e^{-q^2(1+w^2)}$ and $\beta = 1 - \mathrm{erf}(q)\mathrm{erf}(u)$ are extremely small numbers for all reasonable values of parameters p, q, w. The absolute difference of knot efficiency statistical properties and its solid approximation partners depends mostly on some linear combination of α and β. This makes solid approximation so accurate and worthy of attention.

Theorem 6. *Let $\delta^\dagger$ and $\Delta^\dagger$ be absolute PDF and CDF error proposed by Definition 12. Then*

- *absolute difference of PDF functions is bounded for any $\forall\eta, \mu \in \mathbb{R}$:*

$$|\psi(\eta) - \psi^\dagger(\eta)| \leq \frac{1}{r}\delta^\dagger(\eta/r) \qquad , \qquad |\phi(\mu) - \phi^\dagger(\mu)| \leq \delta^\dagger(\mu); \tag{59}$$

- *absolute difference of CDF functions is bounded for any $\forall\eta, \mu \in \mathbb{R}$:*

$$|\Psi(\eta) - \Psi^\dagger(\eta)| = |\Phi(\mu) - \Phi^\dagger(\mu)| \leq \Delta^\dagger(\mu) \leq \alpha + \beta. \tag{60}$$

Before we proceed to the proof, it is important to stress that both $\delta^\dagger(\mu)$ and $\Delta^\dagger(\mu)$ converges rapidly to zero with increasing parameter q. For example, if $p = 4$ and $q = 8$ (hence $w = 0.5$) then $\Delta^\dagger(\infty) = \alpha + \beta = 1.81 \cdot 10^{-35} + 1.54 \cdot 10^{-8} \approx 1.54 \cdot 10^{-8}$. This means that there is practically no observable difference between relative knot efficiency CDF $\Phi(\mu)$, and its solid approximation $\Phi^\dagger(\mu)$ for $\forall\vec{v} \in \Xi$.

Proof of Theorem 6. Proof requires less algebraic manipulations when carried out for ϕ branch of Theorem 6. The ψ branch would be proven similarly. According to Definitions 8 and 11, function $\phi(\mu)$ can be divided into terms containing $\phi^\dagger(\mu)$ and the rest: $\phi(\mu) = \frac{1}{\pi}\frac{w\mathrm{e}^{-q^2(1+w^2)}}{1+w^2\mu^2} + \phi^{\dagger}(\mu)\mathrm{erf}[s(\mu)]\mathrm{erf}[q]$. Then the difference $|\phi(\mu) - \phi^{\dagger}(\mu)|$ can be rewritten in terms: $|\phi(\mu) - \phi^{\dagger}(\mu)| = \left|\frac{1}{\pi}\frac{w\mathrm{e}^{-q^2(1+w^2)}}{1+w^2\mu^2} + \phi^{\dagger}(\mu)(\mathrm{erf}[s(\mu)]\mathrm{erf}[q] - 1)\right|$. Using the triangle inequality [34], previous expression can be upperbounded:

$$|\phi(\mu) - \phi^{\dagger}(\mu)| \leq \frac{1}{\pi}\frac{w\mathrm{e}^{-q^2(1+w^2)}}{1+w^2\mu^2} + \phi^{\dagger}(\mu)|\mathrm{erf}[s(\mu)]\mathrm{erf}[q] - 1|. \tag{61}$$

Assuming $\min\{p,q\} \leq s(\mu)$ and $\mathrm{erf}(t)$ is nondecreasing function we can go further and set upper bound even higher:

$$|\phi(\mu) - \phi^{\dagger}(\mu)| \leq \frac{1}{\pi}\frac{w\mathrm{e}^{-q^2(1+w^2)}}{1+w^2\mu^2} + \phi^{\dagger}(\mu)(1 - \mathrm{erf}[u]\mathrm{erf}[q]) = \delta^{\dagger}(\mu). \tag{62}$$

Based upon the same definitions formula for absolute difference of CDFs can be written: $|\Phi(\mu) - \Phi^{\dagger}(\mu)| = \left|\int_{-\infty}^{\mu}\phi(t)\mathrm{d}t - \int_{-\infty}^{\mu}\phi^{\dagger}(t)\mathrm{d}t\right| = \int_{-\infty}^{\mu}|\phi(t) - \phi^{\dagger}(t)|\mathrm{d}t \leq \int_{-\infty}^{\mu}\delta^{\dagger}(t)\mathrm{d}t$. It is easy to prove that $\int_{-\infty}^{\mu}\frac{\alpha}{\pi}\frac{w}{1+w^2t^2}\mathrm{d}t = \alpha\left[\frac{1}{2} + \frac{\arctan(w\mu)}{\pi}\right]$ and $\int_{-\infty}^{\mu}\phi^{\dagger}(t)\mathrm{d}t = \Phi^{\dagger}(\mu)$ by definition. Hence $|\Phi(\mu) - \Phi^{\dagger}(\mu)| \leq \Delta^{\dagger}(\mu)$.

Assuming that both $\arctan(w\mu)$ and $\Phi^{\dagger}(\mu)$ are non-decreasing functions, we can go further and make the upper bound simpler yet even higher:

$$|\Phi(\mu) - \Phi^{\dagger}(\mu)| \leq \Delta^{\dagger}(\mu) \leq \lim_{\mu\to\infty}\Delta^{\dagger}(\mu) = \mathrm{e}^{-q^2(1+w^2)} + [1 - \mathrm{erf}(q)\mathrm{erf}(u)] = \alpha + \beta. \tag{63}$$

□

Theorem 7. *Let $\bar{\eta}_{[0,1]}$ be the expected value of knot efficiency on interval $\eta \in [0,1]$ (see Definition 9) and $\bar{\eta}^{\dagger}_{[0,1]}$ is the solid approximation of the same quantity (see Equation (53)). Let $\varepsilon^{\dagger}_{\bar{\eta}}$ be the absolute mean error from Definition 12. Finally, let vector (p,q,r) point to sector $\Xi \subset \Pi$. Then, the absolute difference of both means is guaranteed to be lower than:*

$$\left|\bar{\eta}_{[0,1]} - \bar{\eta}^{\dagger}_{[0,1]}\right| \leq \frac{\varepsilon^{\dagger}_{\bar{\eta}}}{\Psi(1) - \Psi(0)} \approx \varepsilon^{\dagger}_{\bar{\eta}}. \tag{64}$$

Proof of Theorem 7. Absolute difference of both means can be rewritten in terms of $\psi(\mu)$:

$$\left|\bar{\eta}_{[0,1]} - \bar{\eta}^{\dagger}_{[0,1]}\right| = \left|\frac{\int_0^1 \eta\psi(\eta)\mathrm{d}\eta}{\Psi(1) - \Psi(0)} - \frac{\int_0^1 \eta\psi^{\dagger}(\eta)\mathrm{d}\eta}{\Psi^{\dagger}(1) - \Psi^{\dagger}(0)}\right| = \int_0^1 \eta\left|\frac{\psi(\eta)}{A} - \frac{\psi^{\dagger}(\eta)}{B}\right|\mathrm{d}\eta. \tag{65}$$

For the sake of simplicity new symbols $A = \Psi(1) - \Psi(0)$ and $B = \Psi^{\dagger}(1) - \Psi^{\dagger}(0)$ have been introduced. Integrand can be rearranged and bounded using the triangle inequality: $\left|\frac{\psi}{A} - \frac{\psi^{\dagger}}{B}\right| = \left|\frac{1}{A}(\psi - \psi^{\dagger}) + \left(\frac{1}{A} - \frac{1}{B}\right)\psi^{\dagger}\right| \leq \frac{1}{A}|\psi - \psi^{\dagger}| + \left|\frac{1}{A} - \frac{1}{B}\right|\psi^{\dagger} \leq \frac{1}{Ar}\delta^{\dagger}(\eta/r) + \frac{|B-A|}{AB}\psi^{\dagger}$. Hence, the integral problem has been changed in following manner:

$$\left|\bar{\eta}_{[0,1]} - \bar{\eta}^{\dagger}_{[0,1]}\right| \leq \frac{1}{A}\left(\int_0^1 \frac{\eta}{r}\delta^{\dagger}(\eta/r)\mathrm{d}\eta + |B - A|\int_0^1 \eta\frac{\psi^{\dagger}(\eta)}{B}\mathrm{d}\eta\right). \tag{66}$$

The remaining absolute value $|B - A| = |A - B|$ can be rewritten using triangle inequality: $|A - B| = |\Psi(1) - \Psi(0) - \Psi^{\dagger}(1) - \Psi^{\dagger}(0)| \leq \underbrace{|\Psi(1) - \Psi^{\dagger}(1)|}_{\Delta^{\dagger}(1)} + \underbrace{|\Psi(0) - \Psi^{\dagger}(0)|}_{\Delta^{\dagger}(0)}$.

Now, relation $\Delta^{\dagger} \leq \alpha + \beta$ from Theorem 6 will be used to simplify the expression and set an upper bound: $|A - B| \leq \Delta^{\dagger}(1) + \Delta^{\dagger}(0) \leq \alpha + \beta + \alpha + \beta = 2(\alpha + \beta)$. Hence the second integral upper limit is:

$$|B - A| \int_0^1 \eta \frac{\psi^{\dagger}(\eta)}{B} \mathrm{d}\eta \leq 2(\alpha + \beta) \bar{\eta}^{\dagger}_{[0,1]}. \tag{67}$$

The remaining unsolved integral can be rewritten using Definition 12:

$$\int_0^1 \frac{\eta}{r} \delta^{\dagger}(\eta/r) \mathrm{d}\eta = \frac{\alpha}{\pi} \underbrace{\int_0^1 \frac{\eta}{r} \frac{w}{1 + w^2 \frac{\eta^2}{r^2}} \mathrm{d}\eta}_{\frac{1}{2} \frac{\ln(1+w^2/r^2)}{w/r}} + \beta B \underbrace{\int_0^1 \eta \frac{\psi^{\dagger}(\eta)}{B} \mathrm{d}\eta}_{\bar{\eta}^{\dagger}_{[0,1]}}. \tag{68}$$

Sum all the terms together and get:

$$\left| \bar{\eta}_{[0,1]} - \bar{\eta}^{\dagger}_{[0,1]} \right| \leq \frac{1}{A} \left(\frac{\alpha}{2\pi} \frac{\ln(1 + w^2/r^2)}{w/r} + [2\alpha + 2\beta + B\beta] \bar{\eta}^{\dagger}_{[0,1]} \right). \tag{69}$$

Now we can finish the proof assuming $B \leq 1$ and $\bar{\eta}^{\dagger}_{[0,1]} \leq 1$:

$$\left| \bar{\eta}_{[0,1]} - \bar{\eta}^{\dagger}_{[0,1]} \right| \leq \frac{1}{A} \left(\frac{\alpha}{2\pi} \frac{\ln(1 + w^2/r^2)}{w/r} + 2\alpha + 3\beta \right) = \frac{\varepsilon^{\dagger}_{\bar{\eta}}}{\Psi(1) - \Psi(0)}. \tag{70}$$

□

Theorem 8. *Let $\sigma^2_{\eta[0,1]}$ be the knot efficiency variance on interval $\eta \in [0,1]$ (see Definition 9) and $\sigma^{2\dagger}_{\eta[0,1]}$ be the solid approximation of the same quantity (see Equation (54)). Let $\varepsilon^{\dagger}_{\sigma^2_{\eta}}$ be the absolute variance error from Definition 12 and let vector (p, q, r) point to sector $\Xi \subset \Pi$. Then absolute difference of both variances is guaranteed to be lower than:*

$$\left| \sigma^2_{\eta[0,1]} - \sigma^{2\dagger}_{\eta[0,1]} \right| \leq \frac{\varepsilon^{\dagger}_{\sigma^2_{\eta}}}{\Psi(1) - \Psi(0)} \approx \varepsilon^{\dagger}_{\sigma^2_{\eta}}. \tag{71}$$

Proof of the Theorem 8 follows the same logical strategy as the Proof of Theorem 7. However, at the same time, it is a very complex chain of algebraic manipulations, basic calculus, and triangular inequality application. It may be disruptive to the flow of the main text, so the proof has been left to the reader.

Finally, it should be acknowledged that there are certainly many another approximations to the knot efficiency distribution. For example, the *normal approximation* is among the most common. This kind of approximation was already studied in papers [10,24,32,35–37]. It is described by normal distribution $\psi(\eta) \approx \hat{\psi}(\eta) \sim \mathcal{N}(\hat{\eta}, \hat{\sigma}^2)$, where $\hat{\eta} = \frac{\bar{x}}{\bar{y}}$ and $\hat{\sigma}^2 = \frac{\bar{x}^2}{\bar{y}^2} \left(\frac{\sigma^2}{\bar{x}^2} + \frac{\varsigma^2}{\bar{y}^2} \right) = \frac{r^2}{2} \left(\frac{1}{p^2} + \frac{1}{q^2} \right)$. It is quite comfortable and user-friendly to work with $\hat{\psi}$ as it is symmetric and Riemann-integrable. It has finite mean and variance. Mode, median, and mean are the same number, and tolerance intervals can be simply evaluated. However, normal approximation is a much too radical simplification to describe knot efficiency in general. Perfectly bell-shaped normal distribution fits the knot efficiency PDF satisfactorily only within specified interval centered at $\hat{\eta}$ (for more details see [37]) and it is not sufficiently accurate for arbitrary $(p, q, r) \in \Xi$.

3. Discussion

In order to make the presented results more comprehensible, some illustrative examples are calculated and discussed. The figure eight loop is one of the most widely used single-loop knots so it is not surprising that it has been chosen as a representative example.

The only difference between the two examples is the degree of rope damage due to its professional mission.

3.1. Figure Eight Loop Knot, Geometry O, Old Worn Rope

Let us suppose that the normal distribution model fits the breaking strength of knotted rope $X \sim \mathcal{N}(\bar{x}, \sigma^2)$ as well as straight rope $Y \sim \mathcal{N}(\bar{y}, \varsigma^2)$. Population means $\bar{x}, \bar{y}$ and variances σ^2, ς^2 are known, and summarised in Table 3, and particular probability density functions are plotted in Figure 6.

Table 3. Parameters $\bar{x}, \bar{y}, \sigma, \varsigma$.

$\bar{x}[kN]$	$\bar{y}[kN]$	$\sigma[kN]$	$\varsigma[kN]$
21.4975	28.1589	1.12131	2.76045

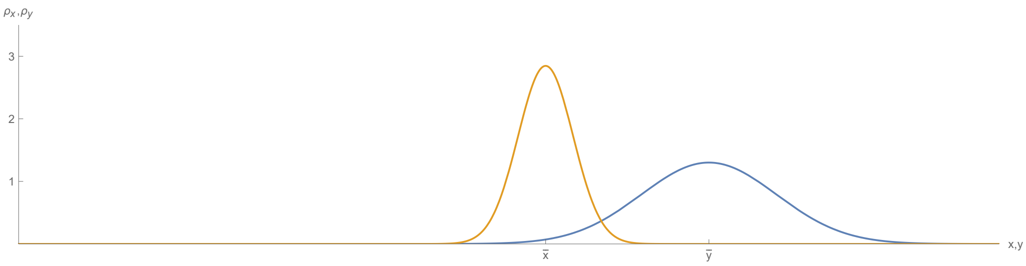

Figure 6. Probability density functions of knotted rope breaking strength ρ_X (yellow) and straight rope breaking strength ρ_Y (blue).

- Parameters p, q, r, w are calculated according to Definitions 6 and 8:

$$p = \frac{\bar{x}}{\sqrt{2}\sigma} = 13.56 \quad , \quad q = \frac{\bar{y}}{\sqrt{2}\varsigma} = 7.21 \quad , \quad r = \frac{\bar{x}}{\bar{y}} = 0.76 \quad , \quad w = \frac{p}{q} = \frac{\bar{x}\varsigma}{\bar{y}\sigma} = 1.88$$

- PDF function $\psi(\eta)$ and CDF function $\Psi(\eta)$ are way too complicated. Let's check whether using solid approximation counterparts $\psi^\dagger(\eta), \Psi^\dagger(\eta)$ would mean a significant accuracy issue or not.
  For the sake of appropriate accuracy mode selection, parameters α and β have to be evaluated firstly:

$$\alpha = \mathrm{e}^{-q^2(1+w^2)} = 3.89 \cdot 10^{-103} \quad , \quad \beta = 1 - \mathrm{erf}(q)\mathrm{erf}(u) = 1 - \mathrm{erf}^2(q) = 3.93 \cdot 10^{-24}$$

  According to Theorem 6, the absolute difference between CDF function $\Psi(\eta)$ and solid approximation counterpart $\Psi^\dagger(\eta)$ for any $\forall\eta \in \mathbb{R}$ is negligible:

$$|\Psi(\eta) - \Psi^\dagger(\eta)| \leq 3.93 \cdot 10^{-24}.$$

  The same goes for absolute difference of truncated mean and variance calculated using solid approximation:

$$\left|\bar{\eta}_{[0,1]} - \bar{\eta}^\dagger_{[0,1]}\right| \leq 1.18 \cdot 10^{-23} \qquad , \qquad \left|\sigma^2_{\eta[0,1]} - \sigma^{2\dagger}_{\eta[0,1]}\right| \leq 9.04 \cdot 10^{-23}.$$

  Therefore, it is reasonable to use a wide spectrum of solid approximation advantages and keep results highly accurate at the same time.
- Statistical assessment of knot efficiency.
  - *Probability density function*

$$\psi(\eta) \approx \psi^\dagger(\eta) = \frac{1}{\sqrt{\pi}} \frac{17.76}{\mathrm{erf}[7.21]} \frac{1 + 4.63\eta}{(1 + 6.06\eta^2)^{\frac{3}{2}}} \mathrm{e}^{-\frac{315.32(\eta - 0.76)^2}{1 + 6.06\eta^2}};$$

– *Cumulative distribution function*

$$\Psi(\eta) \approx \Psi^{\dagger}(\eta) = \frac{1}{2}\left(1 + \frac{\operatorname{erf}\left[\frac{17.76(\eta-0.76)}{\sqrt{1+6.06\eta^2}}\right]}{\operatorname{erf}[7.21]}\right);$$

– *Median*

$$\tilde{\eta} \approx \tilde{\eta}^{\dagger} = \frac{\tilde{x}}{\tilde{y}} = 0.7634 = 76.34\%;$$

– *Mode*

Solid approximation of knot efficiency PDF is a continuously differentiable function so the mode can be found by solving two conditions $\frac{d\psi^{\dagger}}{d\eta}(\eta_m^{\dagger}) = 0$ and $\frac{d^2\psi^{\dagger}}{d\eta^2}(\eta_m^{\dagger}) < 0$. Numerical methods have been employed.

$$\eta_m \approx \eta_m^{\dagger} = 0.7493 = 74.93\%;$$

– *Truncated mean and variance*

Knot efficiency population does not have a finite mean and variance. However, it is possible to evaluate finite truncated mean and variance restricted to the interval of physically meaningful knot efficiencies $\eta \in [0, 1]$. Integrals (53) and (54) have to be solved by numerical integration.

$$\bar{\eta}_{[0,1]} \approx \bar{\eta}^{\dagger}_{[0,1]} = 0.7674 = 76.74\% \quad , \quad \sigma^2_{\eta[0,1]} \approx \sigma^{2\dagger}_{\eta[0,1]} = 0.0068 = 0.68\%;$$

– *Tolerance intervals*

Assumption $5 \leq q = 7.21$ has been met; therefore, solid approximation tolerance intervals $1\sigma \approx 1\sigma^{\dagger}$ and $2\sigma \approx 2\sigma^{\dagger}$ can be evaluated:

$$[\eta^{\dagger}_{1L}, \eta^{\dagger}_{1H}] = [68.53\%, 85.64\%] \quad , \quad [\eta^{\dagger}_{2L}, \eta^{\dagger}_{2H}] = [61.84\%, 96.95\%].$$

– *PDF graph* (see Figure 7.)

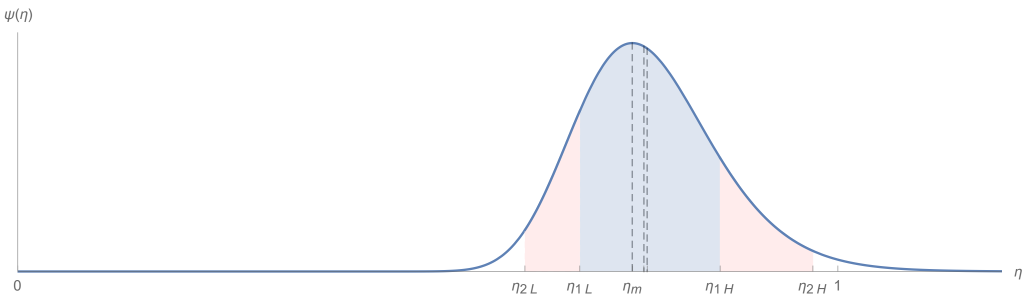

Figure 7. PDF of figure eight loop knot efficiency population, tied in geometry O on a worn rope. Mode, median, and truncated mean are not the same number.

Concluding remarks:

- The most important conclusion is that knot efficiency is certainly not a sharp valued quantity. Truncated standard deviation is $\sigma_{\eta[0,1]} = \sqrt{\sigma^2_{\eta[0,1]}} = 8.25\%$. Tolerance 1σ interval is 17.11% wide and 2σ range is even 35.11% wide. This is one of the reasons why it is so important to make as many breaking strength measurements on knotted and straight rope as possible.

- Niether tolerance intervals nor knot efficiency PDF are not symmetric, with a heavier tail on the right side. Any attempts to approximate this by normal distribution would result in misleading conclusions.
- There is a considerable probability to measure knot efficiency higher than $\eta = 1$. Namely, $P(\eta > 1) = 1 - \Psi^{\dagger}(1) = 1.17\%$. This is not to be confused with truncated mean knot efficiency, median, or mode, which are well below 1 for any real knot.
- Mode, truncated mean and median are not the same number and it is important to decide which of these properties suit user purposes the best.

3.2. *Figure Eight Loop Knot, Geometry O, New Rope*

Let us suppose that the normal distribution model fits the breaking strength of knotted rope $X \sim \mathcal{N}(\bar{x}, \sigma^2)$ and straight rope $Y \sim \mathcal{N}(\bar{y}, \varsigma^2)$. Population means $\bar{x}, \bar{y}$ and variances σ^2, ς^2 are known and summarised in Table 4, and particular probability density functions are plotted in Figure 8.

Table 4. Estimated parameters $\bar{x}, \bar{y}, \sigma, \varsigma$.

$\bar{x}[kN]$	$\bar{y}[kN]$	$\sigma[kN]$	$\varsigma[kN]$
19.2063	27.0889	0.50228	0.78148

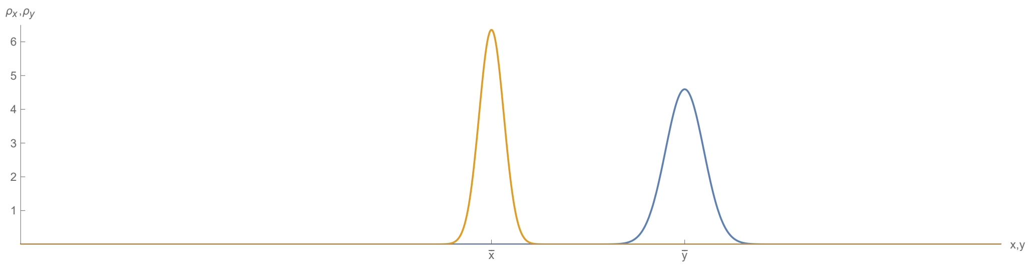

Figure 8. Probability density functions of knotted rope breaking strength ρ_X (yellow) and straight rope breaking strength ρ_Y (blue).

- Parameters p, q, r, w are calculated according to Definitions 6 and 8:

$$p = \frac{\bar{x}}{\sqrt{2}\sigma} = 27.04 \quad , \quad q = \frac{\bar{y}}{\sqrt{2}\varsigma} = 24.51 \quad , \quad r = \frac{\bar{x}}{\bar{y}} = 0.71 \quad , \quad w = \frac{p}{q} = \frac{\bar{x}\varsigma}{\bar{y}\sigma} = 1.10$$

- Let us check whether using solid approximation $\psi^{\dagger}(\eta), \Psi^{\dagger}(\eta)$ would mean a significant accuracy issue or not. For the sake of appropriate accuracy, mode selection parameters α and β have to be evaluated first:

$$\alpha = \mathrm{e}^{-q^2(1+w^2)} = 3.80 \cdot 10^{-579} \quad , \quad \beta = 1 - \mathrm{erf}(q)\mathrm{erf}(u) = 1 - \mathrm{erf}^2(q) = 5.59 \cdot 10^{-263}$$

According to Theorem 6, the absolute difference between CDF function $\Psi(\eta)$ and solid approximation counterpart $\Psi^{\dagger}(\eta)$ for any $\forall \eta \in \mathbb{R}$ is negligible:

$$|\Psi(\eta) - \Psi^{\dagger}(\eta)| \leq 5.59 \cdot 10^{-263}.$$

The same goes for absolute difference of truncated mean and variance calculated using solid approximation:

$$\left|\bar{\eta}_{[0,1]} - \bar{\eta}^{\dagger}_{[0,1]}\right| \leq 1.68 \cdot 10^{-262} \qquad , \qquad \left|\sigma^2_{\eta[0,1]} - \sigma^{2\dagger}_{\eta[0,1]}\right| \leq 1.29 \cdot 10^{-261}.$$

Therefore, it is reasonable to use a wide spectrum of solid approximation advantages and keep the results highly accurate at the same time. Moreover, in this particular case, even normal approximation would be accurate enough for $\eta \in [0, 1]$.

- Statistical assessment of knot efficiency.
  - *Probability density function*

$$\psi(\eta) \approx \psi^{\dagger}(\eta) = \frac{1}{\sqrt{\pi}} \frac{38.14}{\mathrm{erf}[24.51]} \frac{1+1.72\eta}{(1+2.42\eta^2)^{\frac{3}{2}}} \mathrm{e}^{-\frac{1454.33(\eta-0.71)^2}{1+2.42\eta^2}};$$

  - *Cumulative distribution function*

$$\Psi(\eta) \approx \Psi^{\dagger}(\eta) = \frac{1}{2}\left(1 + \frac{\mathrm{erf}\left[\frac{38.14(\eta-0.71)}{\sqrt{1+2.42\eta^2}}\right]}{\mathrm{erf}[24.51]}\right);$$

  - *Median*

$$\tilde{\eta} \approx \tilde{\eta}^{\dagger} = \frac{\bar{x}}{\bar{y}} = 0.7090 = 70.90\%;$$

  - *Mode*
    Solid approximation of knot efficiency PDF is a continuously differentiable function, the mode is a solution to relations $\frac{\mathrm{d}\psi^{\dagger}}{\mathrm{d}\eta}(\eta_m^{\dagger}) = 0$ and $\frac{\mathrm{d}^2\psi^{\dagger}}{\mathrm{d}\eta^2}(\eta_m^{\dagger}) < 0$.

$$\eta_m \approx \eta_m^{\dagger} = 0.7078 = 70.78\%;$$

  - *Truncated mean and variance*

$$\bar{\eta}_{[0,1]} \approx \bar{\eta}^{\dagger}_{[0,1]} = 0.7096 = 70.96\% \quad , \quad \sigma^2_{\eta[0,1]} \approx \sigma^{2\dagger}_{\eta[0,1]} = 0.0007658 = 0.07658\%;$$

  - *Tolerance intervals*
    Assumption $5 \leq q = 24.51$ has been met; therefore, solid approximation tolerance intervals $1\sigma \approx 1\sigma^{\dagger}$ and $2\sigma \approx 2\sigma^{\dagger}$ can be evaluated:

$$[\eta^{\dagger}_{1L}, \eta^{\dagger}_{1H}] = [68.20\%, 73.72\%] \qquad , \qquad [\eta^{\dagger}_{2L}, \eta^{\dagger}_{2H}] = [65.60\%, 76.67\%].$$

  - *PDF graph* (see Figure 9.)

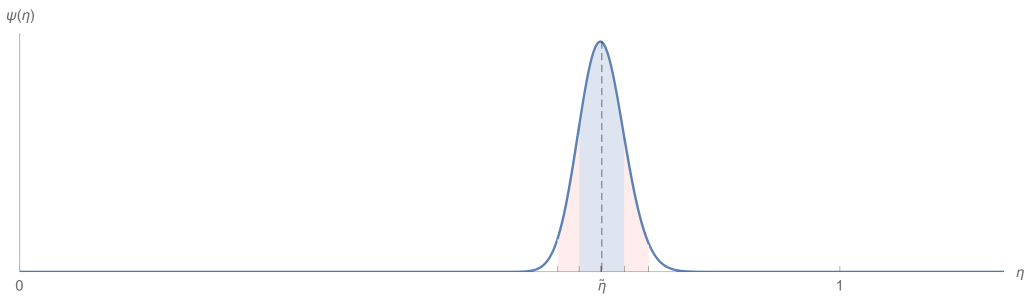

Figure 9. PDF of figure eight loop knot efficiency population, tied in geometry O on a new rope. Mode, median, and truncated mean are effectively indistinguishable.

Concluding remarks:

- The knot efficiency of the same knot tied on the new rope is much better determined than in the previous example. Truncated standard deviation is approximately four times lower: $\sigma_{\eta[0,1]} = \sqrt{\sigma^2_{\eta[0,1]}} = 2.77\%$. However, it is still not a sharp valued quantity.
- Tolerance 1σ interval is 5.53% wide and 2σ range is 11.07% wide. Tolerance intervals and knot efficiency PDF exhibit a high degree of symmetry. Mode, median,

and truncated mean are very close to each other. There is high similarity with a normal distribution;
- Probability to measure knot efficiency from interval $\eta \in \mathbb{R} - [0,1]$ is negligible.
- Normal approximation is much easier to work with and it is recommended to use it if indicated [37]. In such a case mode, median and mean are considered to be the same value: $\hat{\eta}_m = \hat{\tilde{\eta}} = \hat{\bar{\eta}} = r$, variance is given by simple explicit equation: $\hat{\sigma}^2 = \frac{r^2}{2}\left(\frac{1}{p^2} + \frac{1}{q^2}\right)$. Note that the truncated solid approximation variance $\sigma^{2\dagger}_{\eta[0,1]}$ and normal approximation variance $\hat{\sigma}^2$ agree up to 5 decimal places in this particular example.

4. Conclusions

Knot efficiency is a directly immeasurable quantity and the only possibility to obtain this value is calculation by definition $\eta = \frac{X}{Y}$. The vast majority of published results (maybe even all of them) suffer from misunderstanding the statistical nature of knot efficiency as well as incorrect statistical assessment. The main goal of presented paper was to fix this issue. Instead of identifying the knot efficiency with some number, it has to be treated as a random variable with noticeable dispersion, given by a probability density function $\psi(\eta)$ and cumulative distribution function $\Psi(\eta)$. PDF and CDF of knot efficiency have been derived, and all the standard properties of knot efficiency (mean, variance, median, mode, tolerance intervals) have been examined. In addition, a less complex version of knot efficiency (so called solid approximation) has been proposed, and error management was investigated.

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Abbreviations

The following abbreviations are used in this manuscript:

PDF	Probability density function;
CDF	Cumulative distribution function;
$E(X)$	Expected value of the random variable X;
$Var(X)$	Variance of the random variable X;
$\mathcal{N}(\mu, \sigma^2)$	Normal distribution given by the mean μ and the variance σ^2;
$\mathrm{erf}(x)$	Error function: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2} \mathrm{d}t$.

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